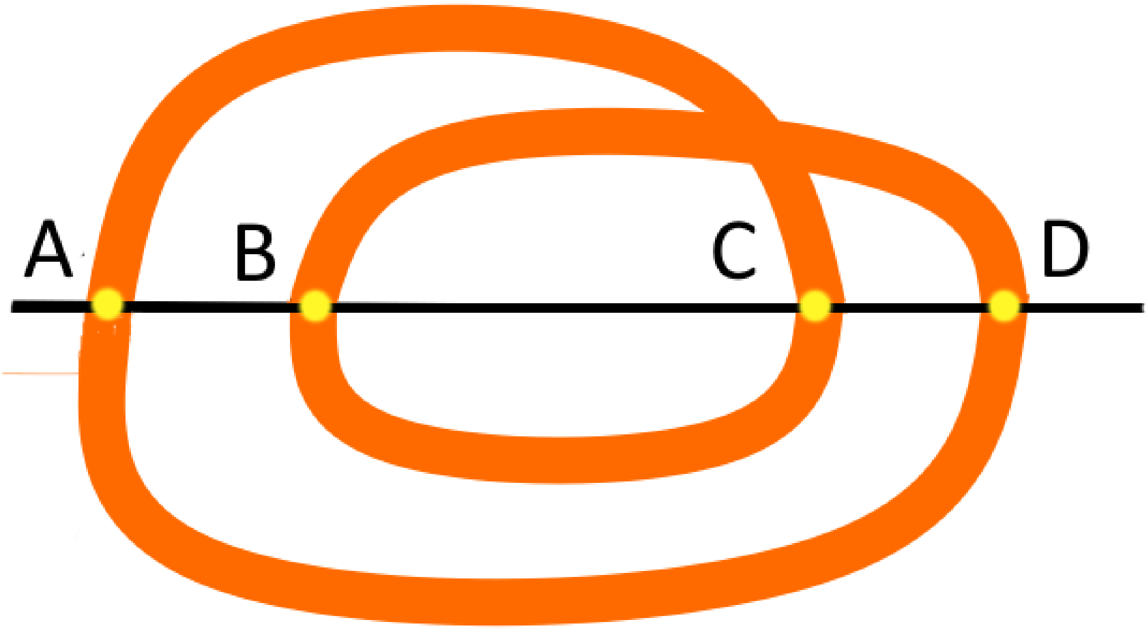


THE CROSS-RATIO

from another point of view

I. **The CROSS-RATIO as a double ratio of the distances of four points belonging to a straight line.**

What I will write in the "Afterword" will only be a brief introduction to the subject, widely treated in projective geometry texts. But in the "game" that I propose in this essay, projective geometry does not concern us. Here I treat the cross-ratio only on the basis of its graphic and algebraic properties, but *without relying on projective geometry.* I only have the modest aim of giving a little more than a mnenonical help to those who have had the same difficulties as me.

A cross-ratio is not formed with any four points on a straight line and making any ratios between the lengths of the segments that have them as ends. No. The points can be chosen at random, and this is true, but the relationships between ratios must be made in a certain well-defined order and *oriented* in a well-defined way.

As a simple example, let A, B, C, D be four equidistant points placed on a straight line in the following way (Fig. 1), and let A be taken as the origin.

Fig.1

As agreed, it is assumed that the distance between two consecutive points is 1. Point A has abscissa 0, point B has abscissa b=1, point C has abscissa c = 2 and point D has abscissa d= 3. We set an orientation of the straight line, such that AB= +1 while BA= -1. The cross-ratio is then given by:

$$(1) \qquad \frac{\frac{CA}{CB}}{\frac{DA}{DB}} = \frac{CA\ DB}{CB\ DA} = \frac{(0-c)}{(b-c)} / \frac{(0-d)}{(0-b)} = \frac{(-2)/(-1)}{(-3)/(-2)} = \frac{4}{3}$$

That is, first of all the four segments that enter into the cross-ratio are written, always putting first the letter that follows in the english alphabet, then the sign is placed. But by what law is the sign placed? Here the choice made is favorable, because the segments, following the order of the written letters, are all negative, but this does not mean that the choice is always so easy. On the other hand, it is not the only one.

Among other things, looking at definition (1), we see that the "cross-ratio" is actually a "double-ratio", or ratio of two ratios. This construction is not exactly intuitive, and, as we will see at the end, it should be a matter of wonder that our eye is somehow able to distinguish it in reality - just as our ear has a particular sensitivity for musical intervals.

I immediately found myself in trouble with the cross-ratio. The formula, first of all, seemed arbitrary and not easy to remember. The arbitrariness of the choice of the four segments and their position as numerator or denominator, and, finally, the confusion of the signs were too many things to remember *without any justification known to me.* Could we put some order into this basic concept of projective geometry?

In the end I think I got there. What I am about to explain has certainly been known to generations of geometry experts, but I had to find it on my own, and I also noticed that a hasty search on various introductory texts of projective geometry finds no mention of the method I used. I only found in some 19th century texts on projective geometry the recommendation to remember the (obvious) diagram:

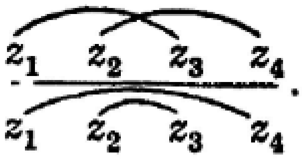


Fig.2

From: *An introduction to projective geometry,* by CW O'Hara (1937)

Let's try to do better (and let those who have already done it forgive me). We will start from an arbitrary straight line l, on which four arbitrary points are marked.

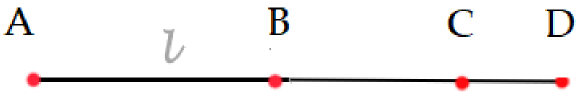


Fig.3

We now join the four points with a **closed curved line** , and we follow it proceeding, in one direction only (without stopping and reversing the direction), starting from any of the points (at first we will begin with A). The closed curved line must *cross* the straight line (or its ideal extension) only once in each of the four points we have chosen.

If we consider the line l as the division between the Northern semiplane and the Southern semiplane, we see that, *starting from A* , and remaining in the Northern semiplane, only six intrinsically different closed curves can start from A. First of all we have a trio of lines: one directed first towards B, one directed first towards C and one directed first towards D. Once the arcs AB, AC, AD have been created, the other two points can be joined in only two ways: that is, we will have ABCD and ABDC; ACBD and ACDB; ADBC and ADCB. We can draw the six variants (Fig.4), and here we notice that curve I is curve VI turned upside down; the II is the IV upside down, the III is the V upside down. Therefore, starting from A towards the Southern half-plane we would find the same six closed curves. **By convention, all the curves will be traversed in a clockwise direction** (with the consequence that, for example, where abscissas are to be dealt with, the abscissa of the departure end will be subtracted from the abscissa of the arrival end.)

But observe again: it is evident that each curve does not change its geometry starting from any of the other three points instead of A. So, for each of the six curves, there are four equivalent ones. Total, 24 curves in all, which are nothing more than the permutations of any four elements A,B ,C,D.

In permutations, any four elements can be, in short, anything. For us they are points ordered on a straight line, each with its own identity card, that is, the abscissa or distance from the origin.

We can therefore think of other cross-ratios, substituting A,B ,C,D in various ways **(but always keeping the order and abscissa of each unchanged)** at the points z_1, z_2, z_3, z_4 of Fig.2. We started with A, B ,C, D. But we could also use the sequence B,A,C,D from which, following the rules of Fig.2, we would obtain the ratio $\frac{CBDA}{DBAC}$ that is different from $\frac{CADB}{CBDA}$. (Here, by chance, the relationship is the inverse of that of Fig.2, but, as we shall see, the relationship could also be more complicated.)

However, we continue with our six closed, neatly composed curves

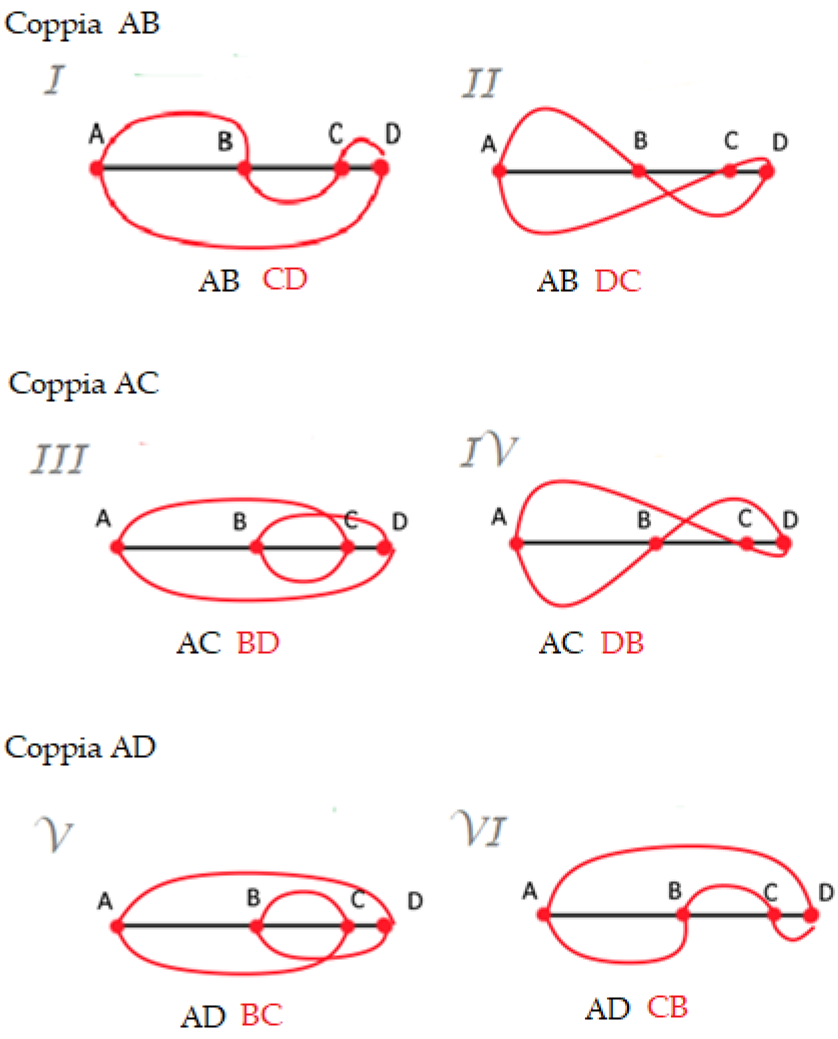
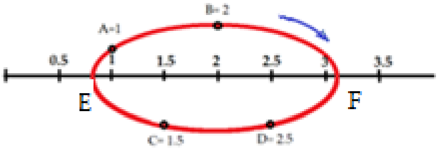


Fig.4

How can we transform these curves into cross-ratios? Magically.

Let's take l as the fraction line. The curve arcs that are above the line , *each starting and ending at two points (E,F) on the line,* are the numerator factors (oriented according to the clockwise reading of the entire curve), and the arcs below the line will be the denominator factors. As mentioned, the line is considered to be traveled in one direction only. See the following example



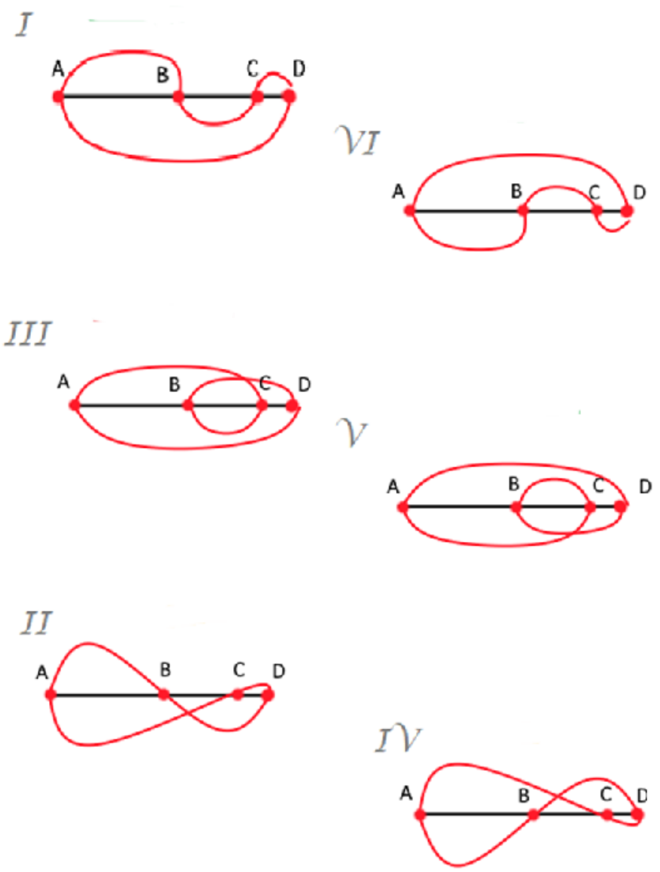
$$= \frac{EF}{FE} = \frac{2-1}{1.5-2.5} = -1$$

Fig. 5

If the ellipse is traversed clockwise, the “ratio” equals -1. But this is the same result if the ellipse is traversed counterclockwise, because both the numerator and the denominator change sign.

Let's reorder our curves, keeping in mind that, based on the above, by turning upside down a curve, the numerator and the denominator change place, and therefore the value of the curve becomes its inverse.

We get like this:



$$\frac{AB\ CD}{BC\ DA}$$

$$\frac{AD\ CB}{DC\ BA}$$

$$\frac{AC\ BD}{CB\ DA}$$

$$\frac{AD\ BC}{DB\ CA}$$

$$\frac{AB\ DC}{CA\ BD}$$

$$\frac{AC\ DB}{BA\ CD}$$

Fig.5

As one can easily observe, the directions of the arcs (which is the first and which is the second letter) can be exchanged as long as the exchanges occur in pairs, since one exchange of extremes changes the sign of the arc, but the change of two signs leaves the result unchanged.

But what exactly have we done? The cross-ratio of four points is a single number and we have found six, or rather 24. What is the true cross-ratio to be used in projective geometry? The answer is simple: following Fig.2 and placing z_1, z_2, z_3, z_4 as the ABCD base permutation, the result is the cross-ratio $\frac{CADB}{CBDA}$ and, given the appropriate symmetries and reading directions, it is the transcription of curve III , which for us is $\frac{ACBD}{CBDA}$. In fact, the denominators are equal and the numerator factors both have the opposite order of the extremes, i.e. the sign.

WE might prefer having the basic traditional cross ratio as our curve I instead of III, but there is little we can do: if we keep the original points in the traditional order A, B, C, D, the cross-ratios also follow in the traditional order.

TABLE I, of the "classic" cross-ratios

(in round brackets)

$$\begin{aligned}
\mathrm{b}(A,B,C,D) = \mathrm{b}(B,A,D,C) = \mathrm{b}(C,D,A,B) = \mathrm{b}(D,C,B,A) &= k,\\
\mathrm{b}(A,B,D,C) = \mathrm{b}(C,D,B,A) = \mathrm{b}(B,A,C,D) = \mathrm{b}(D,C,A,B) &= \frac{1}{k},\\
\mathrm{b}(A,C,B,D) = \mathrm{b}(D,B,C,A) = \mathrm{b}(C,A,D,B) = \mathrm{b}(B,D,A,C) &= 1-k,\\
\mathrm{b}(A,D,B,C) = \mathrm{b}(C,B,D,A) = \mathrm{b}(D,A,C,B) = \mathrm{b}(B,C,A,D) &= \frac{k-1}{k},\\
\mathrm{b}(A,D,C,B) = \mathrm{b}(B,C,D,A) = \mathrm{b}(D,A,B,C) = \mathrm{b}(C,B,A,D) &= \frac{k}{k-1},\\
\mathrm{b}(A,C,D,B) = \mathrm{b}(B,D,C,A) = \mathrm{b}(C,A,B,D) = \mathrm{b}(D,B,A,C) &= \frac{1}{1-k}.
\end{aligned}$$

Following our rules, on the other hand, we obtain Table II (reading the points clockwise and closing our cross-ratios in square brackets). The rows of table I are then messed up (as can be seen from the values of the cross-ratios, which we shall calculate, shown on the right in red).

TABLE II (from reading our closed curves clockwise)

(in square brackets)

(I) [ABCD] = [BCDA] = [CDAB] = [DABC] =m (unknown for now. See below) **1-k**

(VI) [ADCB]= [BADC] = [CBAD]= [DCBA] = 1/m (since the figure is simply upside down.)$\frac{1}{1-k}$

(II) [ABDC]= [BDCA]= [DCAB]= [CABD] = n (or 1/n, unknown for now)$\frac{k-1}{k}$

(IV) [ACDB] = [CDBA]= [DBAC]= [BACD] = 1/n (on, see above)$\frac{k}{k-1}$

(III) [ACBD]=[CBDA] = [BDAC] = [DACB] **=** k (classic name of this relationship)

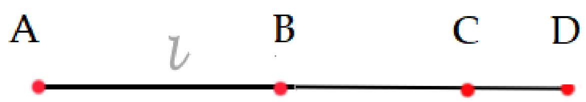
(V) [ADBC]= [DBCA] =[BCAD] = [CADB] = $1/k$

In order to construct Table I it is rather laborious to list the four cross-ratios that have the same value. Instead, in Table II, given above, the permutations within each row are cyclical (as can be seen by reading the same curve in the same direction), starting from the initial one, whose first letter is A. In Table II, however, the permutations are not listed according to initials, as is traditionally done in Table I.

Probably, if the cross-ratio had originated as a representation of a group of permutations of four elements, an ordering like the one I will use would have been preferred. The fact is that the cross-ratio was born from geometric considerations, and the various permutations were only studied later.

II. NUMBERS

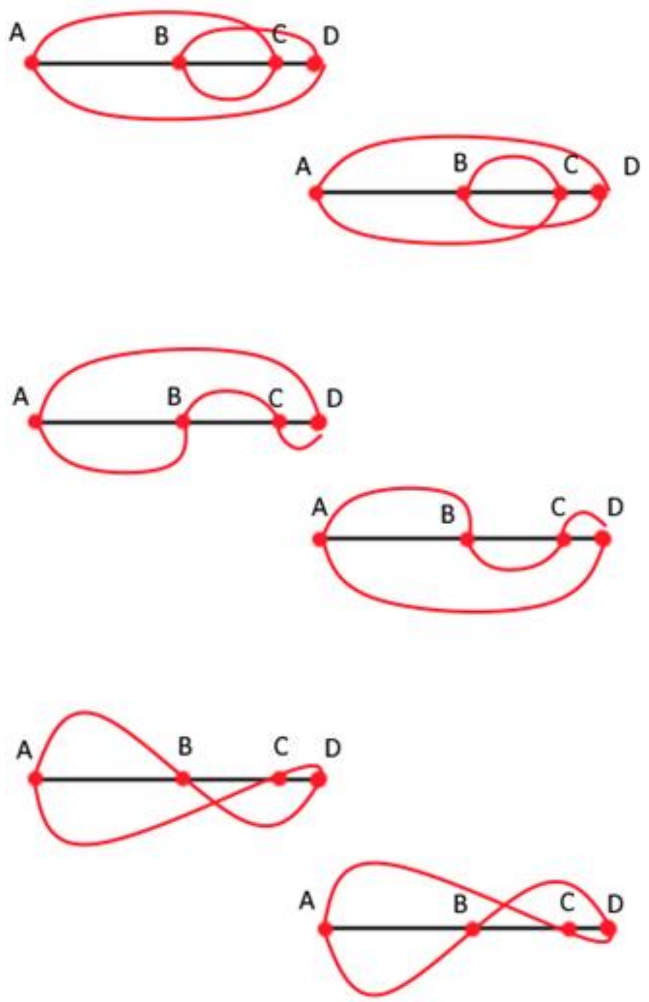
Our task reduces to calculating the values of m and n in Table II, as a function of k, to verify the values given in Table II.
Let's take Fig.3 as a starting point:



It takes no time to realize that we have the freedom to choose one of the points (here A) as the origin, which we will call O, and to assign the abscissa = 1, to B, which we will hereafter call "1", so that its abscissa (AB = O1) defines the unit of length of our system. Two points remain, to which we will give the names of X and Y, with abscissae x and y. With this semialgebraic approach, we can see that all the general properties that we will find by exploiting the choice 0,1 ,X,Y will hold for any group of four points on a line.

If we assigned a value other than 1 to the segment A B, we would naturally have to scale the other abscissae so as to make them coherent with AB, but then we would just introduce complications which at this point would not lead, I hope, to anything new.

A=0, B=1, C=x, D=y



$$\frac{x(y-1)}{-y(1-x)} = \frac{x(y-1)}{y(x-1)} \quad \text{(III)}$$

$$\frac{y((x-1)}{x(y-1)} \quad \text{(V)}$$

$$\frac{y\ (1-x)}{(x-y)\ (-1)} = \frac{y(x-1)}{(x-y)} \quad \text{(VI)}$$

$$\frac{(x-y)}{y(x-1)} \quad \text{(I)}$$

$$\frac{1\ (y-x)}{(y-1)(-x)} = \frac{(y-x)}{x(y-1)} \quad \text{(II)}$$

$$\frac{x(y-1)}{(y-x)} \quad \text{(IV)}$$

Fig.6

To recognize the relationships between the 6 permutations, it is useful to transform the segment ratios into algebraic expressions. Since these are simple fractions, we can look for those that have the same denominator. The sum of the numerators then becomes the most obvious choice to find simple relationships. And it is another remarkable fact that the denominators are equal two by two.

For example, let's add (I) and (III). The sum of the numerators is: $x(y-1) + x - y = xy - y = y(x-1)$.But thus the numerator is equal to the denominator, so we have (I) +(III) = 1, or rather (I)= 1- k. Hence m = 1- k . (VI) is its inverse, i.e. (VI) = 1/(1-k).

On the other hand, (VI), changing the sign of numerator and denominator, takes on the same denominator as (IV). The sum of the numerators is now: $y(x-1) + x(1-y) = xy - y + x - xy = x - y$. Here too the numerator becomes equal to the denominator, so we have 1-(VI) = (IV). Therefore, (IV) has the value 1 – 1/(1-k) = k/(k-1)

Finally (V) is its inverse, i.e. = (k-1)/k = 1-1/k.

Thus have found six values k, 1/k, 1-1/k, k/(k-1), 1-k, 1/(1-k). If only the two operations "do the inverse" and "subtract from 1" are carried out on the six values just given, one after the other, the result is still and always one of the members of the group, the *cross-ratio group*, which group theory scholars will deal with.

Note on the harmonic quatern

It still remains to be said that if a quatern A, B, C, D has a cross-ratio k = - 1, it is called a **harmonic quatern** , where the cross-ratio is the classical one (the first of Table I, and III of Table II), that is

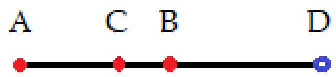
$$\frac{CADB}{CBDA} \equiv \frac{ACBD}{CBDA} = -1$$

The four quaterns of the first row therefore have the same cross-ratio k= -1. But, given that the quaterns of the second row in Table I, or of the last row of Table II have the value 1/k, this means that their cross-ratio is also worth 1/k = -1, i.e. they are also harmonic quaterns. *In other words, there exists a total of eight harmonic quaterns.*

What peculiarities does a harmonic quatern have?

The official answer is that C and D divide the segment AB internally and externally in the same ratio. Unfortunately, also in this case few texts stop to explain what the phrase "they divide externally" means.

However, noting that the cross-ratio $\frac{ACBD}{CBDA} = -1$ tells us that $\frac{AC}{CB} = \frac{-DA}{BD}$and observing the figure



with abscissa $x_A = 0$, $x_C = 2/3$, $x_B = 1$, $x_D = 2$, we see that AC/CB= 2, and AD/BD =2 (where AD has replaced – DA.) To be so, the abscissa of D must be 2. Other than using the formula accompanied by the diagram, I don't see many ways to explain what "split externally" means.

Using our abscissae (0, 1, x, y), which, as has been said, can be applied to any quatern, we have $\frac{x(y-1)}{y(x-1)} = -1$ or $xy - x = -xy + y$ or rather $2xy = x + y$, which illustrates the symmetry of the quatern. In fact, the equation can be solved both in terms of x and in terms of y . It results:

$$y = \frac{x}{2x-1}; x = \frac{y}{2y-1}$$

Normally, the first three terms of the ratio are given, and the unknown is y. In our example, therefore, $y \equiv x_D = \frac{\frac{2}{3}}{\frac{4}{3}-1} = 2$, as the figure shows. If we know y=2 and look for x, we find x= 2/(4-1)= 2/3.

AS an example, the use of the harmonic quatern can explain to us the name "harmonic". It comes from the fact that if you try to calculate the fourth harmonic, given 0, ½ and 1, you get 1/3. To achieve this, remember that we have now replaced the abscissa 1 with the abscissa ½, and therefore, by readjusting the abscissas, we will first have to calculate the harmonic conjugate of 0,1,2. Our formula produces x = 2/3, which however must still be divided by 2 to go back to the original abscissas, finally producing 1/3.
The next step is to find the harmonic conjugate of the first three numbers, which are now (0, 1/3, ½). The harmonic conjugate will be (1/3) of our ratio resulting from (0, 1, 3/2), i.e. 1/3 (3/2 /(3-1)) = ¼. Etc. If we continue to do this operation, taking the first three terms each time (the next triple is 0, 1/4, 1/3), and adjusting their abscissae, we subsequently obtain the terms of the so-called harmonic series (1, ½, 1/3, 1/4 ….)
Of course, the harmonic quatern has several other applications.

III. AFTERWORD

Projective geometry was praised by many, including the physicist Dirac, as one of the most elegant branches of mathematics. One can agree. But, at the University of Turin, where I arrived by mistake, parachuted from the Classical High School, it was a nightmare for me, also because the relatively intuitive discussion of analytic geometry was preceded by a much more abstract introduction on projective geometry.

The introduction began with a brief discussion of Felix Klein's " **Erlangen Programme** " (1872), which stated that the various branches of geometry, which had been developing since the end of the 18th century, could be classified on the basis of *projective geometry* (as a unifying concept) and to *group theory* (as a tool, as the various branches of geometry each studied a different *group of transformations* of the geometric objects they treated.)

> To summarize the program in more detail, it is difficult to do better than Wikipedia, to which I refer: (https://it.wikipedia.org/wiki/Programma_di_Erlangen)
>
> At a more basic level, the text by *Courant and Robbins "What is mathematics", Chapter IV, 1950 edition* is still valid .

All such transformations have in common the fact that in every geometry there exist invariant elements specific to it. For example, the transformations of **Euclidean geometry** preserve the angles and lengths of the objects they deal with, and therefore are limited to "transformations" which are translations and rotations. **"Affine" geometry,** on the other hand, does not care about angles and lengths, but limits itself to studying alignments of points, parallelism, incidence (also invariant in Euclidean geometry), bridging the space between Euclidean geometry and projective geometry.

What about the **projective geometry** ?

Our text placed the feared "**Cross-ratio**" at the basis of *projective* geometry . The cross-ratio, as we have seen, refers to the distances between four points chosen at will on any straight line, and to a double ratio of the distances between pairs of points taken in a certain order.

Historically, the cross-ratio was studied very early, and is accompanied by several theorems of illustrious mathematicians of the past, such as Girard **Desargues (**1591-1661), perhaps the founder of projective geometry, who introduced the concept (but not the name) of cross-ratio. Colin **MacLaurin** (1698-1746) and Jean-Victor **Poncelet** (1788-1867) contributed largely. Poncelet introduced the name " **birapport** "

In fact, its true importance lies in the fact that the cross-ratio is invariant under projections, which makes it an important basic element of projective geometry: this was first considered by Lazare **Carnot** (1803), normally busy with completely other matters, while August Ferdinand **Moebius** (1790-1868) is credited with demonstrating that the cross-ratio can be considered the only invariant of projective geometry . Möbius demonstrated that, regardless of how the points are projected or transformed within the projective plane, as long as they remain collinear, the cross-ratio will remain the same.

Surprisingly, the concept of cross-ratio was not unknown to the Greeks.

The invariance of the cross-ratio is ultimately what allows us to recognize correspondences between photographs and reality. For example, the cross-ratio does not vary if we, having chosen a straight road on which at least four equidistant poles are planted, compare it to a photograph of it taken from an angle. The poles depicted in the photograph appear to be closer and closer to each other the further they are from the observer, and the more foreshortened the photograph is taken. The getting closer to each other of the images of the poles, however, is not arbitrary, but follows a law that the human eye recognizes. The law is that the cross-ratio of the distances of the images of the four poles (measured on a photograph) is equal to the cross-ratio that would be calculated with the real distances.

Such non-obvious property of perspectives was noticed by the Greeks, and the mathematician **Pappus of Alexandria** discovered it ,or at least made use of it, in Book VII of his *Collection* (c.340 AD). But how and why the Greeks arrived there, although they did not have cameras and did not know the rules of perspective, which were invented in the Italian Renaissance (I would cite Filippo Brunelleschi as one, if not the first, of the inventors), remains a small mystery.