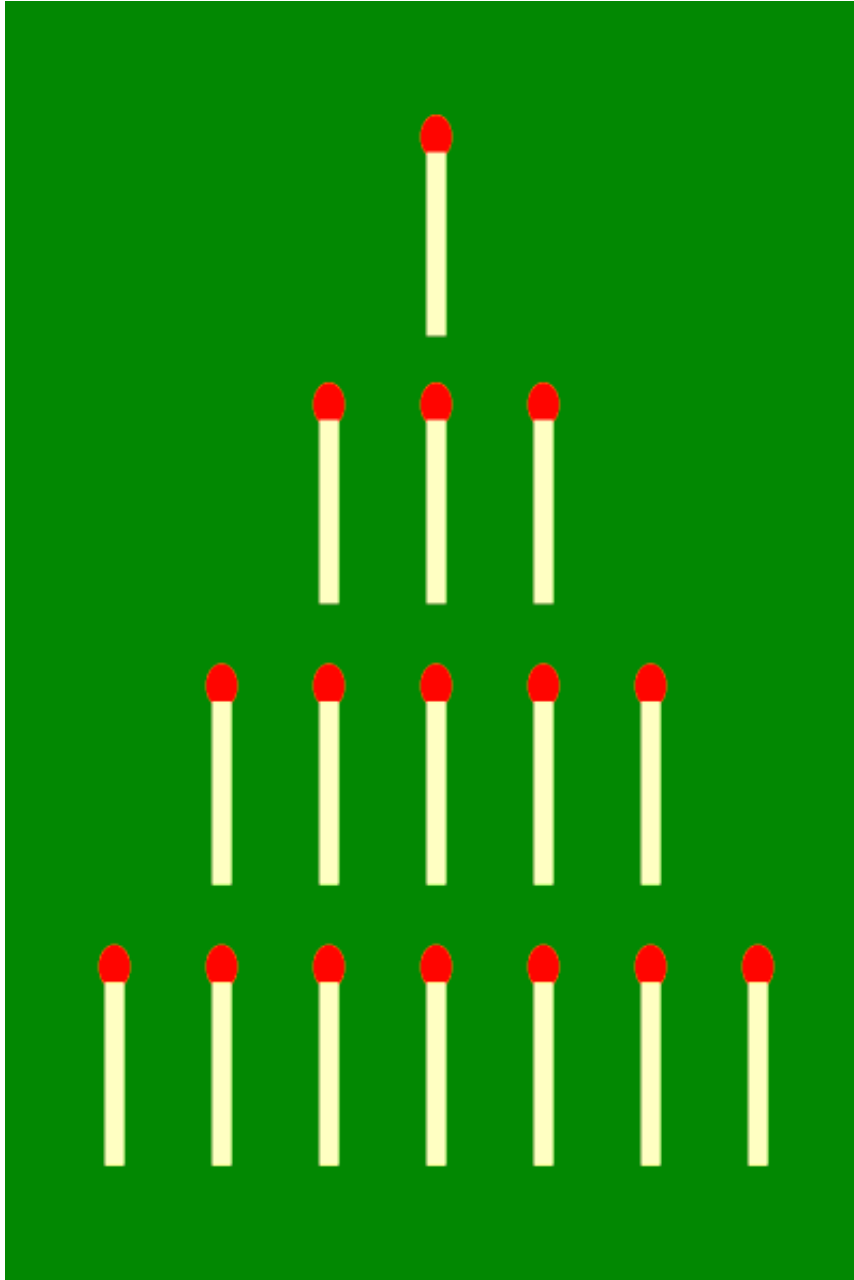


NIM



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An elementary introduction for pedestrian mathematicians

DE

Winter 2022, third year of the Covid Era

INTRODUCTION

The present essay is the (slightly) expanded version of an answer to the question:

How does one realize that XOR is needed to solve the game of Nim?,

which appeared on Quora, english version, years ago. The interest on Nim does not seem to be very high, but my approach is that questions on Quora are just pretexts for me to study some problems I have heard about years ago, without ever having the time to look at them in depth. The story of Nim, for example, goes back to the Seventies.

The simple answer is that XOR as a logical operator is not needed to solve the game of Nim. In fact Prof. Bouton, the inventor of Nim (1) never used the logical operator XOR in his only paper, which laid the foundations of the mathematical theory of the game, but reduced the problem to the use of the **sum modulo 2**, a concept which comes rather from modular arithmetics (Gauss) or from the theory of finite (Galois) fields of order 2.

$\mathbb{F}_2$:

+	0	1	×	0	1
0	0	1	0	0	0
1	1	0	1	0	1

Tab.1

Sum and multiplication in the Galois Field $\mathbb{F}_2$ – the multiplication is not used in the demonstration of the winning strategy for Nim.

The problem with most explanations of the game given on popular texts (including most Wikipedia's), is that one receives the impression that the **XOR (exclusive OR)** and **Mex (Minimum excluded)** concepts somewhat magically give a solution, and there is no explanation how this rather abstruse approach was reached. Besides, it gives the impression that, without possessing at least an introductory notion of Boolean algebra, it is impossible to win ad Nim.

Bouton, who in 1901 was the first to explain the Nim winning strategy did not use such concepts.

I will now try to figure out how Bouton reached his conclusions, which form the basis of the game. Sadly, Prof.C.L. Bouton could not talk to me, and what follows is a sort of "Mathematical Fiction" exercise, but, of course, the reader is free to accept or reject my reconstruction.

Definition of the game (from <https://en.wikipedia.org/wiki/Nim>)

Nim is a mathematical game of strategy in which two players take turns removing (or "nimming") objects from distinct heaps or piles. On each turn, a player must remove at least one object, and may remove any number of objects provided they all come from the same heap or pile. Depending on the version being played, the goal of the game is either to avoid taking the last object or to take the last object.

The first option "He who takes the last object wins " is called "the **normal** nim game"; the second option "He who takes the last object loses", is called the "**misère**" variant of the nim game. I will concentrate on explaining the genesis of the winning strategy for the "normal game". Once one understands the principles leading to a normal game winning strategy, then there exists a simple method to convert it into a winning strategy for the "misère" variant.

I will call each act of removing objects as a "**remove**" (a noun, which exists in English, although is not very used. Still, I like it, because it sounds like "move", the typical unit of action in many games).

1. One pile.

Let's suppose we have the simplest form of Nim: one pile of objects (I like to think of **coins**).

To play the "one pile" game we must change the rule given above by Wikipedia, otherwise, the first player can take the whole pile, and invariably wins. Instead, **players can only remove 3 or 2 or 1 coins**.

To put some order in our mind, let's align the coins in a row. The first player (A) removes the first 1, or 2, or 3 coins. Then the second player (B) does the same, and the two players take turns, until they reach the end of the line: the player who in the last remove grabs the last coin wins.

Safe and unsafe positions.

It takes just a short reflection to see that there are "winning" or "safe" positions, and "losing" or "unsafe" positions. What do the terms "safe" and "unsafe" mean? The property of the safe positions is that by taking 3, 2, 1 coins one cannot reach the next safe position, **otherwise the opponent could do it**. The property of the unsafe positions is that it has always (at least) one way to reach a safe position.

The last winning position before victory is the *fourth place from the end*. If a player (A) hands over the game to B with only four coins left, then B has lost, because he is obliged to take at least one coin, which enables player A to take the last three coins, **including the last coin**. On the other hand B cannot take 4 coins. Thus we can say that a position is safe for the player who reaches it. A safe position, on the other hand, is a position of doom for the player who has to take action from a safe position. But we see that for A to reach that safe position, he had to start from the safe position in the *eighth place from the end*. In conclusion, a position, to be safe, is at a multiple of four places from the last.



Fig.2

The Nim game with a single pile of 13 coins (and two players). Starting from the left, and alternatively removing either 1 or 2 or 3 coins, the player who can remove the last coin on the right wins.

This is not really a game like chess or checkers, and as soon as the two players discover the trick, the game becomes kind of boring. The conclusion is that, if the initial number of coins is randomly decided, in one case out of four, on average, the first player, A, has to deal with a safe position. In that case, if B knows the game, A has lost, and there is no way to correct the situation. On the other hand, in the three cases out of four in which he does not start from a safe position, the first player wins, because he can remove enough coins to reach a safe position, and his opponent cannot do anything about it. Thus the advice is: **try to be the first player**. The players probably get bored soon, but at least they learn two concepts, that of **safe position**, which needs two removes to reach another safe position, and that of **unsafe position**, which only needs one.

Consequently, the pattern of a correct game is the following:

0. Player A sets up a safe position

Move 1: B's turn, cannot remove to a safe position, makes any possible remove, the total number of coins decreases.

Move 2: **A sets up a safe position in one remove** from where B stopped, the total number of coins decreases.

Move 3: **B's turn, cannot remove to a safe position**, makes any possible remove, the total number of coins decreases.

Move 4: **A sets up a safe position in one remove** from where B stopped the total number of coins decreases.

....

Move 2n: **A reaches** the winning position. The total number of coins is 0.

Such pattern is the basis of the winning strategy of all Nim-like games.

2. Two piles

Let's now see what happens when we have two piles of coins. Here, we go back to the rule as given by Wikipedia: one player can take *out of one of the piles as many coins as he wants, even the whole pile*. Taking a whole pile is not a good idea, because the opponent wins by taking the whole other pile.

The final winning position is when we have **1 coin left in both piles**. Suppose A has reached or built that position. Then B has lost, because he must take one coin, i.e. he must empty one pile, and A then can win by emptying the other pile.

Which are the safe positions both players must try to reach? Obviously, those in which both piles have the same number of coins. The point is that, if the A player has left any of such configurations on the table, the B player can take coins only from one pile, and by so doing he perturbs the equality of coins in the two piles. The A player can then restore the equality (and therefore reach a *safe position*) by removing the *same number of coins* from the other pile.



Fig.3

Two piles Nim Game. The “safe “ or “winning” positions are those in which one player sets up two piles with the same number of coins.

3. Three piles.

Let's finally consider the more interesting case of three piles, which can easily be extended to n piles. Let's suppose that we do not put more than 10 coins in each pile.

3.1 Safe positions

Let's finally consider the more interesting case of **three** piles, which can easily be extended to n piles. Let's suppose that we do not put more than 10 coins in each pile.

3.1 Safe positions

a) Final safe position: $(0,1,1)$ (in all examples the piles can be permuted with no change of the position. Thus, $(0,1,1)=(0,1,0)=(1,1,0)$.)

We start again from the end of the game. The player who sets up the position $(0,1,1)$ is safe, because his opponent must take one coin, leaving the final one to the initial player, who thus wins - in the “normal” game.

b) Second safe position (from the end) : $(1,2,3)$

Now we look for the preceding safe position, which must be two removes away from the $(0,1,1)$ or from any other safe position.

It cannot be $(1,1,1)$, because with one single remove the opponent can reach the final safe position $(0,1,1)$; as a general rule one position is not safe if it contains two equal numbers of coins in one pile. In fact, if player A sets up the (n,n,m) position, player B, in one remove, gets to the safe position $(n,n,0)$ we have already met in Section 2, by removing the third pile. The game continues as the “two pile game” we have considered above, and stays so, because one cannot rebuild the third pile (players can only remove coins, they cannot add them) - and A is in trouble. Thus, the only possibility of a safe position for player A is $(1,2,3)$.

We see that player (B), if confronted with the $(1,2,3)$ position, can take the 1 from the first pile. Then player A takes 1 from the third pile: the position is now $(0,2,2)$ a safe position, which, at worst, will become $(0,1,1)$ in the next turn. If B, instead, takes coins leaving a position $(1,1,n)$ or equivalent, A takes the whole non-1 pile and is back to $(0,1,1)$, or equivalent. Etc.

The above example suggests a general link between nim games with different numbers of piles: a position in the **4-pile** game is not safe if it includes three piles with numbers of coins matching a safe position of the **3-pile** game. The player, whose turn it is, can remove the last pile, and lands on a safe position of the 3-piles game. Thus, in the four pile game we do not expect to see, for example, any safe position of the kind “1, 2, 3, n”, because the player who must play could remove the fourth pile in a single move and play the 4-pile game as if it were a 3-pile game, starting from a safe position.

More in general we can say: **Rule I: a safe position of an n-pile game, cannot have n-1 piles forming a safe position in the n-1 piles game.**

c) Third safe position (from the end): $(1,4,5)$

What is the next closest safe position moving backwards in the game?

We can now give **Rule II: Not only the player must avoid having two piles with the same number of coins, but two different safe positions cannot have two piles with the same numbers of coins, and a third with a larger number of coins**, because, if that is the situation, the opponent can move from one safe position to another safe position *with one remove*. Thus, in our case, the new position must not have two numbers equal to two

numbers among 1, 2, 3. Suppose we keep 1 at the first place. Then the second number cannot be either 2 or 3, and the smallest allowed number will be 4. The third number cannot be either 1 (Rule I) or 2 or 3 or 4 (Rule II), and therefore will be 5. Thus, the next safe position is **1, 4, 5**.

Lemma of Rule II: in a safe position the sizes of two piles determine in a unique way the size of the third pile.

d) Fourth safe position: (1,6,7) and other (1,n,m) positions

The next position after (1,4,5), is obtained still keeping 1 and by considering again that the second number cannot be 1 (rule I). By rule II, it cannot be either 2, or 3 (because then in a single move we fall back on 123), and cannot be either 4 or 5 (because we fall back on 1,4,5). It must be 6, and the last is 7.

Thus, (1, 6, 7), is the fourth safe position.

Then we will have **(1,8,9), fifth safe position**, always applying Rule I and Rule II. The series of safe positions with 1 in the first place ends here, if we keep the number of coins at a maximum of 10 in any pile (if we remove the limit, then (1,10,11) would be next, etc.) Also, we note that 1 will not appear in any future safe positions, as long as 10 is the maximum number of coins.

e) To go to the next safe position, we must have a 2 in the first place. The next pile cannot have either 2 coins by rule I, or 1 or 3 by rule 2. It can be 4. In the third place we cannot have 1 (fully used), nor 2 (rule I), nor 3 (close to 1,2,3), nor 4 (rule I), nor 5 (Rule II, applied to 1,4,5). We must have 6.

Sixth safe position, (2,4,6).

The next safe position has 2 coins in its first place, and in the second place there cannot be either 2 (rule I), or 1, or 3, or 4 (rule II). We can have 5. Again application of the rules we have given to the third place, excludes 5 and 6, and therefore the **seventh safe position** is **(2,5,7)**.

It does not take a mathematical genius to continue, and discover that next we have **(2,8,10)**, as (2,8,9) is excluded by Rule II, applied to (1,8,9).

Then we have **(3,4,7), (3, 5, 6)** and **(3, 9, 10)**.

We have no other safe positions, if we limit ourselves to a maximum of 10 coins per pile.

Positions			Total
0	1	1	2
1	2	3	6
1	4	5	10
1	6	7	14
1	8	9	18
2	4	6	12
2	5	7	14
2	8	10	20
3	4	7	14
3	5	6	14
3	9	10	22

Table 2.

All safe (or winning) positions, with maximum number of coins =10 for each pile.

To win a game with 3 piles and a maximum of 10 coins each, this is all we need. We can learn by heart the safe positions (no extraordinary feat), or write them on a paper. However, for some reason, people do not refuse to play with a person who learned by heart the 10 or 11 positions, but refuse to play if one reads them out of a slip of paper.

C. Bouton must have asked himself **what was common to all safe positions**, to make them stand out as the key to the solution of the Nim game.

Clearly, the **decimal expression** did not tell much. The sums of the three numbers, on the other end, resulted in a first interesting result: by summing the amounts of coins in the piles of all safe positions, both in the two columns and 3 columns cases we obtained an even number.

Then he must have thought that, if one wants to go into the machinery of a number, one must resort either to the prime factorization, or to the binary basis, because, as a rule, it is difficult to see in any base what you cannot see in base 2.

The **prime factorization** does not tell much: four times we have a total of 14 coins. Now, $14 = 2 \times 7$, where 7 is an odd prime, but 12 and 20 do not fit into this picture. At first sight, there is no obvious common feature resulting from the prime factorization.

I believe that, perhaps after other attempts, C. Bouton turned to the **binary basis**, which allowed him to compile the following table, which is nothing but the translation of Table 2 to binary basis:

			0 0 0		
			0 0 1		
			0 0 1		
			-		
0 0 1	0 0 1	0 0 1	0 0 0 1	0 1 0	0 1 0
0 1 0	1 0 0	1 1 0	1 0 0 0	1 0 0	1 0 1
0 1 1	1 0 1	1 1 1	1 0 0 1	1 1 0	1 1 1
a	b	c		d	e
	0 0 1 0	0 1 1	0 1 1	0 0 1 1	
	1 0 0 0	1 0 0	1 0 1	1 0 0 1	
	1 0 1 0	1 1 1	1 1 0	1 0 1 0	
		f	g		

Table 3

All *safe position* for piles not exceeding 9 coins. The positions marked as a,b,c etc. are those with piles not exceeding 7 coins, excluding the (0,1,1) and (0, 0, 0) piles.

To these, the nine positions of the form (0, n, n) – including (0,1,1) - should also be added.

I would bet that after writing all the eleven safe positions in binary form, our man Bouton immediately realised what was common to all of them, namely that every column included only either all zeros or one pair of 1's (considering that there are only three rows). In addition, he must have realised that two rows of a triplet determine the third row, if one wants to make the triplet fall into the common pattern (a fact we already know: **3.1.c Lemma**). In binary notation the same result becomes obvious, because the third row is necessary and sufficient, to compensate for the missing 1's or 0's. Thus, to reach a safe position, one does not need to go through a comparison with all the known safe positions, smaller than the starting set. We know by the rules of the game, that only one pile (here represented in a row) can be modified, in particular reduced.

Now we come to the core of my answer: the sum modulo 2 has the same table as the XOR logical operator of Boolean algebra. However, in no demonstration of the winning strategy for Nim one really needs to use the properties of XOR in some sort of logical proposition.

x	y	$x \oplus y$
0	0	0
1	0	1
0	1	1
1	1	0

Table 4

Table of the results of acting with the XOR operator on two numbers.
Basically, it is the same as the “+” table of Fig.1

As we said, Bouton never mentioned more than the “**sum modulo 2**”, a concept which comes rather from modular arithmetics (Gauss) or from the theory of finite fields (Galois) of order 2 (Bouton was an expert in advanced group theory.)

In section 2 of his article, he immediately tells us that one safe position is found by writing on three rows the three number of coins in each pile in binary form, and aligning the columns. Then, sort of casually, he says: “**If the sum of each column is 2 or 0 (i.e. congruent to 0 mod.2) the set of numbers form a safe position.**” He does not give any explanation of such an extraordinary prescription, and therefore I believe that he arrived at it as I showed, by empirically constructing the set of consecutive safe positions, starting from the end of the game, putting them in binary form, and examining them. The property he mentions leaps to the eye, and Bouton wrote as if he was examining Table 3, which he had built - but without telling us how.

3.2 The number of safe positions.

One last point needed a demonstration: that **all and only all** the positions with sums by column all equal to zero, using the mod.2 sum, were accounted for by an empirical construction like that, which I have given in Section 3.1.

One answer to the given question is to calculate the total number of safe positions both in a theoretical and in an empirical way. The results must be the same. Let’s see how the total number of safe positions can be calculated, in the simplest possible way. For example let’s verify that **7 is the number of safe position for piles not exceeding 7 coins**. If one does so with a maximum of 7 coins for each pile, he will have to fill a 3 x 3 square matrix with zeros and ones.

The number is restricted because a column can give a mod 2 sum equal to zero only if it assumes one of 4 forms.

0	1	0	1
0	1	1	0
0	0	1	1

Table 5.

The 4 possible columns whose sum Mod2 (bitwise, i.e. without carries) of the elements is 0.

Each of the four columns can combine with any others of the quartet. The binary forms of the numbers up to 7 require only three columns, and therefore we can have up to 4^3 combinations, that is a total of 64 positions. However, we can subtract as less interesting the positions of the form (0,n,n), seven of them, and their permutations (we can put the 0

in any of the three places), for a total of $7 \times 3 = 21$ positions (which were treated apart). Such operation brings us down to 43 combination. We then subtract, as even less interesting, the “all zeros” position. We are now at 42 positions. But looking now at the rows, (that is, looking at the numbers of coins in each pile), each position has six equivalent permutations (i.e., numbering the **rows** as 1,2,3, we have the permutations 123,132, 213, 231, 312, 321, all equivalent for Nim). Thus we have $42/6$ intrinsically different positions, i.e. a grand total of 7, which tallies with Table 1. **The different safe positions which contain up to 7 coins are the seven positions labelled a,b,c,d,e,f,g, plus seven positions of the form (0, n, n) or equivalents.**

In the same manner, a table empirically constructed by Bouton, with up to and including 15 coins per pile, produces **35 safe positions**, and a calculation along the lines we followed above gives the same number.

Doing the calculation is easy if we consider tables for positions, in which all piles have at most $2^n - 1$ coins. For example, we can calculate the safe positions for a maximum of 15 objects ($= 2^4 - 1$) on **three piles**. Then the total number of combination is $4^4 = 256$. We must subtract the $15 \times 3 (= 45)$ combinations, which include an empty pile. This leaves us with 211 combinations. We subtract the unique “all zero” position, and get 210. But each position (or line) appears 6 times (by permuting the lines), which means that the total number of different safe positions for $n = 16$ is $210/6 = 35$, as given by Bouton in his article. Again, we must add to them the 45 positions (3×15) in which two columns are equal.

1 2 3	2 4 6	3 4 7	4 8 12
1 4 5	2 5 7	3 5 6	4 9 13
1 6 7	2 8 10	3 8 11	4 10 14
1 8 9	2 9 11	3 9 10	4 11 15
1 10 11	2 12 14	3 12 15	
1 12 13	2 13 15	3 13 14	
1 14 15			
	5 8 13	6 8 14	7 8 15
	5 9 12	6 9 15	7 9 14
	5 10 15	6 10 12	7 10 13
	5 11 14	6 11 13	7 11 12

Table 6

The 15 “Nim” safe positions for three piles with at most 15 coins (from Bouton’s paper). The 15×3 positions of the form (0, n, n,) as well as the (0,0,0) position are not included.

Thus, there are no stragglers, which have zero sum for each column and are not safe positions.

At that point Bouton had nothing more than a conjecture. He had to transform it into a theorem or a set of theorems.

3.3 Bouton's Theorems.

Bouton immediately after presenting "a safe position" (Section 2) gives two theorems, **the first one**, that *if A leaves a safe position on the table, then B cannot make any remove that leaves a safe position on the table* (which would put A in trouble). This is almost obvious, on the basis of **Lemma 3.1.c**, because two elements of a safe position determine uniquely the third (assuming that there are no empty piles).

Thus, as B can move coins from one pile only, he is forced to abandon a safe position, whatever (re)move he makes, because two safe positions cannot have the same number of coins in two different piles.

The **second theorem** is that *no matter what changes made B to one column of a safe position left by A, A can always remove coins from one of the remaining two columns to make a safe position*. That we must operate on (that is remove coins from) one of the remaining two piles is obvious, from what we said above. Here, Bouton gives a "limited demonstration", which considers only three piles, and, moreover, assumes that the unsafe position from which A starts has been left there by B, who in turn started from a safe position, left by A. *Both restrictions, as we shall see, are unnecessary....* But he did it first. I will give a more general demonstration of the present second theorem.

Furthermore he says that the same rules apply to a game with more than three piles: "*In this case a safe combination is a set of numbers such that, when written in the binary scale and arranged with the units in the same vertical column, the sum of each column is even, that is 0 mod.2.*" Summing column by column (**which is said to sum "bitwise", basically a binary sum without carry-overs**) mod 2, we have 0 0 0 0. **The sum "bitwise mod2" will be called in the following NimSum[., .., ...]** and the numbers on which the NimSum operates will be contained in the square brackets. "Bitwise mod2" or "without carry-overs" seem to me pleonasms: "binary sum mod2" excludes carry overs, and remains bitwise.

It is worth noting that the NimSum possesses the **commutative** and **associative** properties, and furthermore, as shown in the addition table, **NimSum[x,x]=0**, which can be proved by putting first $x=0$ and then putting $x=1$. However, the result can be applied to any two equal numbers irrespective of the number of binary figures 0's and 1's):

Let's suppose that we want the NimSum[13,13], which gives, in binary notation,

1 1 0 1 +

1 1 0 1

The result (of the binary sum Mod.2 - automatically without carry-overs) is NimSum [13,13] = **0 0 0 0** or simply zero. On the other hand, the sum of two different numbers cannot be zero, and is positive (we do not have any negative numbers in our system).

The example shows that if we put in column two equal binary expression, we always have in each column either two pairs of 1's or two pairs of 0's, and their NimSum is always 0.

3.4 Modifications

In his Section 5, Bouton briefly considers the case of n piles. The "safe positions" have the same properties for all numbers of piles from 2 onwards, and "the induction proof [*of the theorem that the player who first sets up a safe combination can do so at every subsequent play, and will win*] is so direct that it seems unnecessary to give it". Thanks.

In Section 6, Bouton observes that the Nim game can be played also in a version in which the winner is the person which forces the other player to take the last coin (the so-called "misère" version).

I give here my own version of the strategy to win the "Misère" version, which differs a bit from Bouton's approach.

Basically, the strategy changes from the "normal" to the "misère" game only at the end of the game, when one player (for example A) reaches the **(1, 2, 3)** position.

At that point, **B** can reduce 123 to six positions: **12; 121; 122; 113; 13 or 23**.

In the "normal case", the final winning position is (0, 1, 1). In four cases it is immediate to see how A can go to the 011 position. In two cases A can easily reach (0, 2, 2) which we know to be a safe position for the normal game.

In the "Misère" game, the final winning position is (1, 1, 1) or (1, 0, 0) and we can see how A can get to either 111 or (better) 001, from any of the positions 12; 121; 122; 113; 13 or 23. Player A can remove **12** to 1; **121** to 111, **122** to 22, **113** to 111, **13** to 1, **23** to 22.

Note that **22** forces B to 12, or 02, and in both cases, A can either take the last coin ("normal" game) or force B to take the last coin ("misère").

3.5 The "standard explanation"

I will not follow Bouton's approach to demonstrate the winning strategy, because, once the NimSum concept with its immediate properties is introduced, the demonstration is simple, and more general than Bouton's demonstration.

While Bouton starts from a "safe position" of a "3-piles game", which appears without explanation, the standard explanation (which appears for example in en.Wikipedia) does

not mention any “safe positions” at all, and is not based on the fact that we are dealing with three piles. It starts from an arbitrary position, without worrying on how did we get there, and calculates the NimSum S of that position. Then it proves that, if $S = 0$, any change in a single pile results in a position in which the sum S is not zero, while if $S > 0$ (it cannot be less than zero) it can be reduced to $S=0$ by operating on a single pile.

The proof is simple as advertised. To be concrete, we suppose we have three piles, whose **numbers of coins** in a starting position are x_1, x_2, x_3 . Having written the numbers of coins in binary forms, their sum columns by column (mod2), which we called $\text{NimSum}[\dots]$, gives a binary number, S . In other words, $S = \text{NimSum} [x_1, x_2, x_3]$.

Let there be no mistake. The reader should not be astonished if in binary form, $\text{NimSum}[101, 110] = 11$, which in decimal notation becomes $5 + 6 = 3$.

After the player who is confronted with a situation **with any S** (de)moves, the new position is $T = [y_1, y_2, y_3]$, with new values for the number of coins in the piles. Actually, we know that only one column (which we can always permute to the third place) has changed its value x_3 to a new y_3 , and $y_3 < x_3$ (in decimal or binary form) because coins can only be subtracted. In all other piles the numbers are unchanged ($x_1 = y_1, x_2 = y_2$).

We have: $T = \text{NimSum}[0, T]$, but the rules of the NimSum tell us that $\text{NimSum}[x, x] = 0$, or, in our case, $\text{NimSum}[S, S] = 0$.

Therefore $T = \text{NimSum}[S, S, T]$, and, by the asociative property, $T = \text{NimSum}[S, \text{NimSum}[S, T]]$, where

$$\text{NimSum}[S, T] = \text{NimSum}[\text{NimSum}[x_1, x_2, x_3] + \text{NimSum}[y_1, y_2, y_3]].$$

We can rearrange T as

$$T = \text{NimSum}[S, \text{NimSum}[x_1, y_1], \text{NimSum}[x_2, y_2], \text{NimSum}[x_3, y_3]].$$

But, thanks to the fact that $x_1 = y_1$ and $x_2 = y_2$, we finally have

$$T = \text{NimSum}[S, \text{NimSum}[x_3, y_3]] \text{ (the only sum which differs from 0)}.$$

We have now two cases:

I) The starting position was $S=0$, which means that after a remove the resulting position cannot be $T=0$,

II) The starting position was $S>0$, which means that the final position can became (with one appropriate remove) $T=0$.

The entire theorem guarantees that if one player (A) sets up a position with $S = 0$, the opponent (B), because of (I), cannot transform the position into another position with $T=0$. On the other hand player A, thanks to (II), can always move from a position $T>0$, created by B, to a position with $S'=0$, and less coins. The process will continue (T unsafe, S' safe; T' unsafe, S'' safe...with less and less coins, until A will win, because the game is won if he can build the position $(0,0,0)$, with $S(N)=0$, which happens by taking the last coin(s), and concludes the game. *The last statement is obvious, but I thought it was better to write it explicitly (en.Wikipedia does not have the same opinion).*

While (I) is immediate, (II) requires some attention. Let's see it in detail.

If $S \neq 0$, written as a single binary number, it will have a leftmost 1 in a position D (from the right). The fact that there is a 1 in the binary representation of S expresses the fact that the S of the position is not equal to 0. We must therefore look for the pile x_K that has a number of coins with a number 1 in that D position, be it $x_K(D)$, in binary notation. It does not need to be the leftmost bit of x_K . There must be at least one such pile, otherwise the $S(D)$ bit, the sum of the binary figures in the D column, would be zero.

Then we put $y_K = \text{NimSum}[S, x_K]$, which is interpreted as a binary number of coins.

We claim that $y_K < x_K$ (numbers of coins), that is, the K pile has decreased. In fact, all the bits to the left of D are unchanged by the remove from x_K to y_K , while bit D , by Nimsumming it to 1, will decrease from 1 to 0, thereby decreasing the value of y_K by -2^D .

Changes occurring at the right of D will amount at most to $+2^{D-1}$. One can consider, as an example, the binary difference of 1000-0001, (that is 8-1), which produces 7 (111).

It is useful to remember that the binary formulation of a number just indicates which powers of 2 appear in the sum which adds up to the number. For example, suppose that a Zero in the D position from the right indicates that the power 2^D has been eliminated. Suppose now that in all positions following D toward right, until the end of the number, there is a 1, which means that all decreasing powers from $2^{(D-1)}$ to 2^0 are present. Then the number is given by the sum of all powers of 2 up to $D-1$, that is, by the sum $1+2+2^2..+2^{(D-1)} = 2^D-1$, which is less than the power 2^D we have eliminated. We must thus conclude that in any case the number decreases. Even if higher powers exist (and are the untouched), the number with 2^D missing is smaller than the number in which 2^D appears.

The bits to the right of D can be arranged in order to create an $S=0$ situation

In other words, the player can **remove $x_K - y_K$ coins** from pile K (as $x_K - (x_K - y_K) = y_K$). The result is that the new T will be equal to zero:

$$T = \text{NimSum}[S, x_K, y_K] =$$

$\text{NimSum}[(S, xK), \text{NimSum}[S, xK]] = 0, \text{cdd}.$

which means that in one single demove, the first player can bring the game to a safe position.

An example will clarify (I hope) the situation and at the same time show what happens if there are more than one pile whose binary representation has a 1 where the S highest bit is (that is in the position we have called D). I have not seen this particular case dealt with elsewhere, but it is not impossible to find it, at least as an initial position. Let's suppose that we have the three piles **(6, 3, 7), that is (110, 011, 111)**. The case – if both players play well – is possible only as an initial position, which we suppose to be randomly selected, while it is an impossible position in the course of a correct game (*Why? Hint: it is impossible to obtain (6,3,7) from a safe position with more coins*). The NimSum of the three piles is 010. As one can see, the first bit to the left of S is at the place D=2, and all three piles have a 1 as the 2nd bit.

1 1 0

0 1 1

1 1 1

S = 0 1 0

Now, S can become 0 if we put 0 in the place of 1 at the second place (from the right) in any of the binary numbers representing each pile. This can be achieved by NimSumming S to the xK pile (which in this case means any of the three piles). We thus have three possibilities: putting 0 in the middle of the top line (and recalling that **4, 3, 7 = 3, 4, 7**); putting 0 in the middle line (**6, 1, 7 = 1, 6, 7**); putting 0 in the bottom line (**6, 3, 5 = 3, 5, 6**). All three positions we obtain are safe.

The results were achieved by inspection, but we should show that the same result can be achieved by following the rules given above. Let's consider $yK = \text{NimSum}[S, xK]$.

- If we select **K=1**, we have $y1 = \text{NimSum}[010, 110] = 100$, corresponding to **y1= decimal 4**;
- for **K=2** we have $y2 = \text{NimSum}[001, 011] = 001$, corresponding to **y2=decimal 1**;
- for **K=3**, we have $\text{NimSum } y3 = [010, 111] = 101$, corresponding to **y3=decimal 5**.

By subtracting from x1 the difference $x1 - y1 = 6 - 4 = 2$ from the first pile we get the three piles **4, 3, 7**; by subtracting $3 - 1 = 2$ from the second pile we have **6, 1, 7**; by subtracing $7 - 5 = 2$ from the third pile, we have **6, 3, 5**. As above, all of them are safe positions.

Besides practising on the way to reach a safe position from an unsafe position, we have discovered a class of positions, which cannot be reached in the course of a correct game. They can occur as the starting position, however, especially if the starting position is randomly selected, to ensure a fair game.

As we can see, *at no step, the standard demonstration depends on the number of piles*, which can be as many as we please. Always a player landing on an $S=0$ (that is safe) position wins, and also the rules to define and create (if necessary) a safe position remain the same. Of course, both players must play a perfect game.

Conclusion

At the end of his comparatively short life (he died at age 53) Charles Leonard Bouton had to suffer from ill health, family problems, personal grief.

Considering the main field of his research in his early years, I think that the paper on the game of Nim (a name, which was coined by Bouton himself, probably from the German word Nimm, which means “take!” – but he never specified the reasons of his choice) was not too seriously taken by its author. Rather, he must have considered it just as a mathematical amusement.

Yet, while no important contributions of his are mentioned in the annals of differential equation, **his paper “Nim, a game with a complete mathematical theory”(1901) on *Annals of Mathematics Vol. 3, No. 1/4, 1901 - 1902* is today considered to have laid the foundations of the combinatorial game theory**, an entire field of mathematics. Good for him!

As a footnote I add that the interest in the game was revived by a famous film: “L'Année dernière à Marienbad” (released in the US as *Last Year at Marienbad* and in the UK as *Last Year in Marienbad*), a 1961 French-Italian Left Bank film, directed by Alain Resnais, from a screenplay by Alain Robbe-Grillet. There are three characters, and one, named M, continuously beats the main (?) character, X, and says: “I can lose , but I always win”.

Wikipedia comments [I put my comments in square brackets]: The film is famous for its enigmatic narrative structure, in which time and space are fluid, with no certainty over what is happening to the characters, what they are remembering, or what they are imagining. Its dreamlike nature has both fascinated [the critics] and baffled viewers; many have hailed the work as an avant-garde masterpiece, although others[,the majority,]have found it incomprehensible. ... *Marienbad* received an entry in [The Fifty Worst Films of All Time](#),...



Fig.1: Neues Schloss Schleißheim

This is NOT the hotel referred to in the movie “Last year at Marienbad” (but close to it).

NOTE

(1) Charles L. Bouton (Saint Louis (MO) 1869 – Cambridge (MA)1922) was a Professor at the University of Harward, where he was rememberd as an outstanding teacher, and the editor/co-editor of two mathematical journals. He had studied two years in Leipzig (with a Parker fellowship), where he was one of the last students of the Norwegian mathematician Sophus Lie (died on 18 February 1899). He received his PhD in 1898, with the dissertation “Invariants of the General Linear Differential Equations and their Relation to the Theory of Continuous Groups,” Advisor : Sophus Marius Lie.

(<https://www.ams.org/journals/bull/1922-28-03/S0002-9904-1922-03508-2/S0002-9904-1922-03508-2.pdf>)