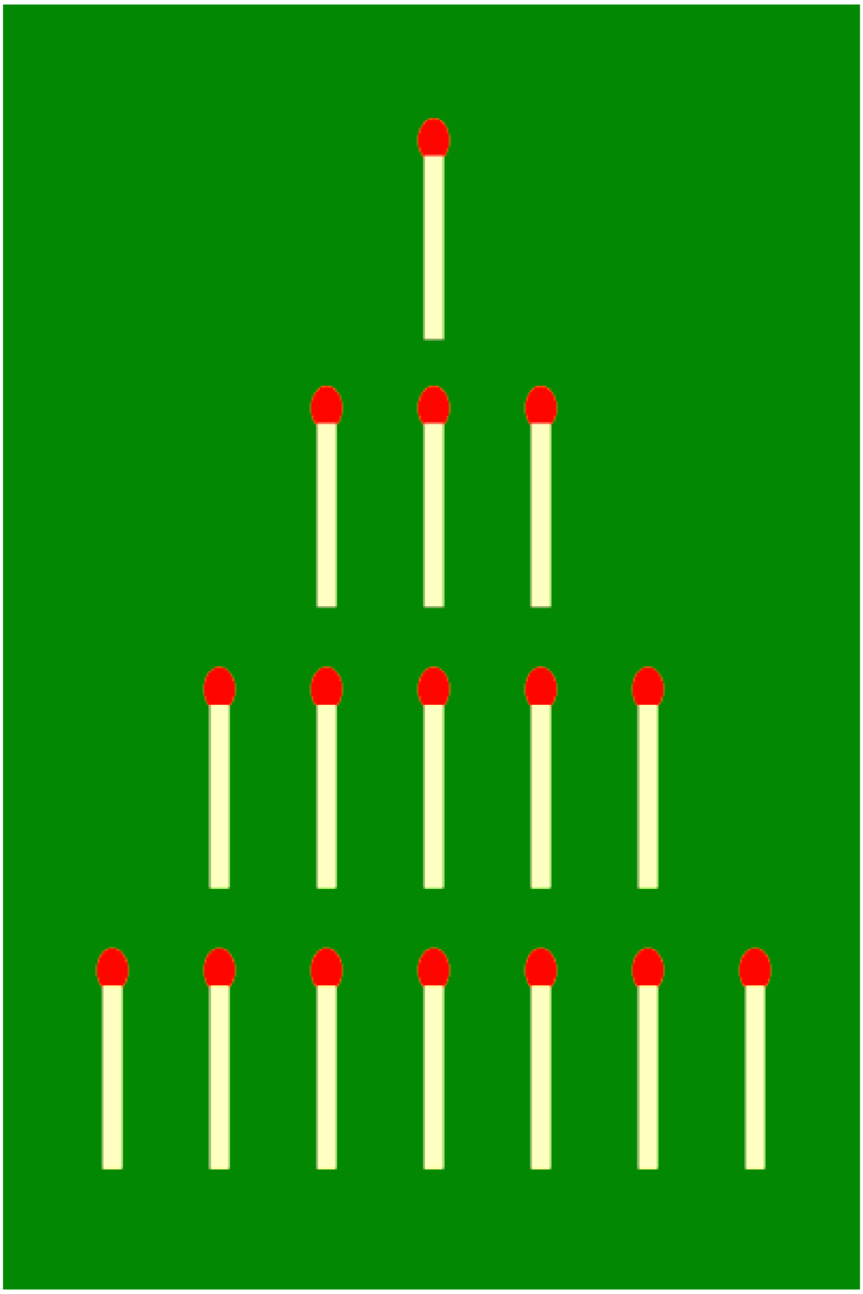
NIM



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An elementary introduction for pedestrian mathematicians

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Winter 2022, the third year of the Covid era

INTRODUCTION

The present essay is the (slightly) expanded version of an answer to the question: **How do you realize that XOR is needed to solve Nim's game ?,** which appeared on Quora, English version, years ago. The interest in Nim doesn't seem to be very high, but my approach is that questions on Quora are just pretexts to study some problems I heard about years ago, without ever having the time to look into them in depth. My history with Nim, for example, dates back to the 1970s.

The simple answer is that XOR as a logical operator is not needed to solve Nim's game. In fact, Prof. Bouton, the inventor of Nim (1) never used the logical operator XOR in his only work, which laid the foundations of the mathematical theory of the game, but reduced the problem to the use of **the modulo sum 2,** a concept that derives rather from modular arithmetic (Gauss) or from the theory of finite fields (Galois) of order 2.

$\mathbb{F}_2$:

+	0	1
0	0	1
1	1	0

×	0	1
0	0	0
1	0	1

Table 1

Sum and multiplication in Galois Field F2 - multiplication is not used in the proof of the winning strategy for Nim.

The problem with many of the explanations of the game given in popular texts (including most of Wikipedia), is that one gets the impression that the concepts **XOR (exclusive OR)** and/or **Mex (minimum excluded)** somehow magically give a solution, and there is no explanation as to how this rather abstruse approach was achieved. Furthermore, it gives the impression that, without having at least an introductory notion of Boolean algebra, it is impossible to win at the game of Nim.

Bouton, who in 1901 was the first to explain Nim's winning strategy, did not use any such concepts.
Now I will try to understand how Bouton came to his conclusions, which are the basis of the game. Unfortunately, Prof. CL Bouton was unable to speak to me, and what follows is a sort of "mathematical science fiction" exercise, but, of course, the reader is free to accept or reject my reconstruction.

Game definition (from https://en.wikipedia.org/wiki/Nim)

***Nim** is a mathematical game strategy in which two players take turns removing objects from distinct piles or piles. At each turn, a player must remove at least one object and may remove any number of objects as long as they all come from the same pile or pile. Depending on the version played, the goal of the game is that of getting the last item or to avoid getting the last item.*

The first option " **Whoever gets the last item wins** " is called "the **normal** nim game"; the second option " **Whoever takes the last item loses " is internationally called the "misère"** variant of the nim game. I will focus on explaining the genesis of the winning strategy for "normal play". Once you understand the principles that lead to a "normal" game winning strategy, there is an easy way of converting it into a "misère" game winning strategy.

I will call any act of removing objects a " **remove** ", rather than "move" or "grab", because the player does not keep the coins.

1. One pile.

Suppose we have the simplest form of Nim: a pile of objects (I like to think of **coins**).

To play the game "with one pile" we have to change the rule given above by Wikipedia, otherwise the first player can take the whole pile, and invariably wins. Instead, **players can only remove 1 or 2 or 3 coins.**

To put some order in our mind, let's line up the coins in a row. The first player (A) removes the first 1, or 2 or 3 coins. Then the second player (B) does the same, and the two players take turns, until they reach the end of the line: the player who in the last remove takes the last coin wins.

Safe and unsafe locations.

A brief reflection is enough to conclude that there are "winning" or "safe" positions and "losing" or "unsafe" positions. What do the terms "safe" and "unsafe" mean? The property of safe positions is that, starting from a safe position and taking 3, 2, 1 coins, one cannot reach the next safe position, **otherwise the opponent could do it.** The property of unsafe positions is that there is always (at least) one way to reach a safe position starting from a different one of them.

The last winning position before the victory is the *fourth place from the end*. If a player (A) gives the game to B with only four coins left, then B has lost, because he is obliged to take at least one coin, which allows player A to take the last three coins, **including the last coin**. B, on the other hand, cannot take 4 coins. So we can say that a position is safe for the player who reaches it with his move. A safe position, on the other hand, is a position of doom for the player to (re) move from a safe position. But we see that to reach that safe position, Player A had to start from the safe position at the *eighth place from the last coin.* In conclusion, a safe position is a multiple of four places from the last.

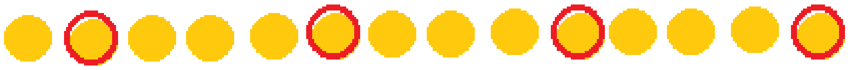

Fig. 2

The Nim game with a single pile of 13 coins (and two players). Starting from the left and alternately removing 1 or 2 or 3 coins, the player who manages to take the last coin on the right wins.

This isn't quite a game like chess or checkers, and as soon as the two players figure out the trick, the game gets a little boring. The bottom line is that, if the initial number of coins is randomly decided, in one out of four cases, on average, the first player, A, has to contend with a safe position. In that case, if B knows the game, A has lost and there is no way to correct the situation. In the three out of four cases, however, since he does not start from a safe position, the first player wins, because he can remove enough coins to reach a safe position, and his opponent cannot do anything about it. So the advice is: **try to be the first player.** Players will likely get bored soon, but at least they learn two concepts, that of **a safe position** , which requires two removes to reach another safe position, and that of **an unsafe position** , which only needs one.

Consequently, the scheme of a correct game is as follows:

0. Player A establishes a safe position

Move 1, B's turn: B by definition cannot move into a safe position . He makes any possible remove, the total number of coins decreases.

Move 2, A's turn: A establishes a safe position in one remove from where B stopped, the total number of coins decreases.

Move 3, B's turn: B by definition cannot move into a safe position . He makes any possible remove, the total number of coins decreases.

Move 2, A's turn: A establishes a safe position in a remove from where B stopped, the total number of coins decreases.

….

Move 2n: A reaches the winning position and takes the last coin. The total number of coins is now 0.

The pattern I presented is the basis of the winning strategy of all Nim-like games.

2. Two piles

Now let's see what happens when we have two piles of coins. Here we return to the rule given by Wikipedia: a player can take as many coins as he wants ***from one* of the piles** , *even the entire pile* . Taking a whole pile is not a good idea, because the opponent wins by removing the entire other pile.

The final winning position is when we have **1 coin left in both piles** . Suppose A has reached or built that position. Then B has lost, because he has to take a coin, that is, he has to clear one pile, and therefore A can win by clearing the other pile, which is the last coin.

What are the safe positions that both players must try to reach? Obviously those where both piles have the same number of coins. The point is, if Player A has left one of these configurations on the table, Player B can only take coins from one pile, and in so doing disrupts the equality of the coins in the two piles. Player A can then restore equality (and thus reach a *safe position*) by removing the *same number of coins from the* other pile.

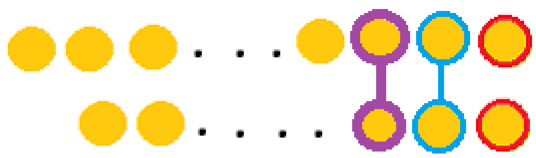

Fig.3

Nim game with two batteries. The "safe" or "winning" positions are those in which a player manages to get two piles with the same number of coins.

3. Three piles.

Finally, let us consider the most interesting case of three piles, which can be easily extended to **n** piles. Suppose we put no more than 10 coins in each pile.

3.1 Safe locations

a) Safe final position: (0,1,1) *(In all examples the piles can be swapped without changing position. Hence, (0,1,1) = (0,1,0) = (1,1,0).)*

We start from the end of the game. The player who sets up the position **(0,1,1)** is "sure" to win, because his opponent has to take a coin, leaving the final one to the starting player, who thus wins - *in the "normal" game.*

b) Second safe position (from the end): **(1,2,3)**

Now we look for the previous safe position, which must be two removes from (0,1,1) or any other safe position.

It cannot be (1,1,1), because with a single remove the opponent can reach the final safe position (0,1,1); as a general rule, a position is not safe if it contains two of the same coins in a pile. In fact, if player A gets position **(n, n, m)** , player B, in a remove, moves to the safe position **(n, n, 0)** that we have already encountered in Section 2, completely removing the third pile . The game continues as the "two-pile game" we considered above, and remains so, because the third pile cannot be rebuilt (players can only remove the coins, they cannot add them) - and A is in trouble. Therefore, the only chance of a safe position for Player A is **(1,2,3).**

We see that player (B), when compared with position (1,2,3), can take the 1 from the first pile. Then player A takes 1 from the third pile: the position is now (0,2,2) a safe position of the two-pile game, which, in the worst case, will become (0,1,1) in the next round. If B, on the other hand, removes coins leaving a position (1,1, n) or equivalent, A takes the entire pile n, and returns to (0,1,1), or equivalent. Etc.

The example given suggests a general link between nim games with different numbers of piles: a position in the **4-piles game** is not safe if it includes three piles with numbers of coins corresponding to a safe position **in the 3-piles game** . The player, whose turn it is, can remove the last pile and land in a safe position of the 3-pile game. So, in the four-piles game we do not expect to see, for example, any safe positions like "1, 2, 3, n", because the player who has to play could remove the fourth pile in one move and play the game with 4 piles as if it were a game with 3 piles, starting from a safe position.

More generally we can say: **Rule I: a safe position in a game with n-piles cannot have n-1 piles which form a safe position in a game with n-1 piles.**

c) Third safe position (from the end): **(1,4,5)**

What is the next closest safe position going back in the game?

We can now give **Rule II** : **not only must the player avoid having two piles with the same number of coins, but two different safe positions cannot have two piles with the same number of coins and a third with a greater number of coins,** because, if this is the situation, the opponent can move from one safe position to another safe position *with a single remove* of coins from a single pile . So, in our case, the new position must not have two numbers equal to two numbers among 1, 2, 3. Suppose we keep 1 in first place. Then the second number can be neither 2 nor 3, and the smallest allowed number will be 4. The third number can be neither 1 (Rule I) nor 2 or 3 or 4 (Rule II), and therefore it will be 5. So, the next safe position is **1, 4, 5.**

Lemma of Rule II: in a safe position, the numbers of coins of two piles uniquely determine the number of coins in the third pile.

d) Fourth safe position: (1,6,7) and other (1, n, m) positions
Always going back in the game, the previous position (1,4,5), is obtained by always keeping 1 and considering again that the second number cannot be 1 (rule I). For rule II, it cannot be either 2 or 3 (because then in a single move we fall back to 1,2 3), and it cannot be 4 or 5 (because we fall back to 1,4,5). It must be 6 and the last is 7.
Hence, **(1, 6, 7), is the fourth safe position.**

Later we will have **(1,8,9)** , **fifth safe position** , always applying Rule I and Rule II. The series of safe positions with 1 in the first place ends here, if we keep the number of coins to a maximum of 10 in any pile (if we remove the limit then (1,10,11) would be the next safe position etc.). , we note that 1 will not appear in any future safe position, as long as 10 is the maximum number of coins.

e) To reach the next safe position, we must have a 2 in the first place. The next pile can have neither 2 coins according to rule I, nor 1 or 3 according to rule 2. It can be 4. Thirdly we cannot have 1 (fully used), nor 2 (rule I), nor 3 (rule II, applied to 1,2,3), nor 4 (rule I), nor 5 (rule II, applied to 1,4,5). We must have 6.
Sixth safe position, (2,4,6).
The next safe position has 2 coins in first place, and in second place there can be neither 2 (rule I), nor 1, nor 3 or 4 (rule II). We can have 5. Again, the application to the third place of the rules we have given excludes 5 and 6, and therefore the **seventh safe position** is **(2,5,7).**

It does not take a mathematical genius to continue, and to discover that we then have **(2,8,10)** , as (2,8,9) is excluded by Rule II, applied to (1,8,9).

Then we have **(3,4,7), (3, 5, 6)** and **(3, 9, 10)** .

There are no other safe positions, as long as we limit ourselves to a maximum of 10 coins per pile

Positions			Total
0	1	1	**2**
1	2	3	**6**
1	4	5	**10**
1	6	7	**14**
1	8	9	**18**
2	4	6	**12**
2	5	7	**14**
2	8	10	**20**
3	4	7	**14**
3	5	6	**14**
3	9	10	**22**

Table 2.
All safe (or winning) positions, with maximum number of chips = 10 for each pile.

To win a game with 3 piles and a maximum of 10 coins each, this is all we need. We can memorize the safe positions (which is not an extraordinary feat), or write them down on a piece of paper. However, for some reason, people don't refuse to play with someone who has memorized the 10 or 11 positions, but they refuse to play if he reads them from a slip of paper.

A brief reasoning based on Rules I and II also convinces us that, playing with **4 piles of coins, the positions** (1, 1, n, n) are safe. Less simple is the case of safe positions (1,3,6,4), (1,2,5,6), (1,2,4,7) (1,3,5,7). (2,3,4,5) (2,3,6,7) (2,3,8,9), (4,5,6,7) (4,5,8,9) (n, n , m, m) (n, n, n, n). - but we will see another way to prove it. The (1, 3, 5, 7) position shown on the cover is safe, and therefore the player, however prudent it may be, if he is the first to play from that position, will certainly lose.

C. Bouton must have wondered **what was common to all safe positions,** to make them stand out as the key to solving the Nim game. Finding an easy "safety" criterion would have facilitated the verification whether they were safe or not.

Clearly, the **decimal expression** didn't say much. The sum of the three numbers, on the other hand, produces an interesting first result, since, by adding the amount of coins in the piles of all safe positions, both in two-column and 3-column games, an even number is obtained.

Then he must have thought that, if one wants to delve into the machinery of a number, one must resort either to the prime factorization, or to the binary basis, because, as a rule, it is difficult to see in other bases what one cannot see. in base 2.
Prime **factorization** doesn't say much: four times we have a total of 14 coins. Now, 14 = 2 x 7, where 7 is an odd prime, but 12 and 20 don't fit in this picture. At first glance, there is no obvious common feature resulting from prime factorization.

I believe that, perhaps after other attempts, C. Bouton turned to **the binary base,** which allowed him to compile the following Table 3, which is nothing more than the translation of Table 2 into binary base (each row giving the number of coins of each pile):

000
001
001
-

001	001	001	0001	010	010
010	100	110	1000	100	101
011	101	111	1001	110	111
a	b	c		d	e

0010	011	011	0011
1000	100	101	1001
1010	111	110	1010
	f	g	

Table 3

All *safe positions* for piles not exceeding 9 coins. The positions marked with a, b, c etc. are those with piles not exceeding 7 coins, excluding piles (0,1,1) and (0, 0, 0).

To these must also be added the nine positions of the form (0, n, n) - inclusive of (0,1,1).

I would bet that, once all eleven **safe positions were written** in binary form, in **line,** our man Bouton immediately realized what was common to all of them, which was that each column included only either all zeros or a pair of 1's (considering that there were only three piles, and therefore three rows). Furthermore, he must have realized that two lines of a triplet determine the third row, if we want to fit the triplet into the common pattern (a fact we already know: **3.1.c Lemma**). In binary notation the same result becomes obvious, because the third line is necessary and sufficient to compensate for missing 1s or 0s. Therefore, in order to reach a safe position, it is not necessary to make a comparison with all known safe positions. We know from the rules of the game, that only one pile (represented here as a **row**) can be modified, or, in fact, reduced.

Now we come to the core of my answer to the question posed on Quora: the sum modulo 2 has the same table as the logical XOR operator of Boolean algebra. However, in no proof of Nim's winning strategy is it imperative to use the properties of XOR in some sort of logical proposition.

x	y	$x \oplus y$
0	**0**	0
1	**0**	1
0	**1**	1
1	**1**	0

Table 4

Table of the results of acting with the XOR operator on the two numbers (0,1). Basically, it is the same as the "+" table in Fig.1

As we said, Bouton never mentioned more advanced mathematical concepts than that of **"sum modulo 2",** a concept that derives rather from modular arithmetic (Gauss, 1801) or from the theory of finite fields (Galois) of order 2 (Bouton was an expert of advanced group theory.)

In section 2 of his article, he immediately tells us that a safe position is found by writing on three lines the number of three coins in each pile in binary form and aligning the columns. Then, more or less casually, he says: " **If the sum of each column is 2 or 0 (ie congruent to 0 mod.2) the set of numbers forms a safe position".** It gives no explanation for such an extraordinary prescription, and so I believe he got there as I have shown, by constructing empirically the set of consecutive safe positions, starting at the end of the game, putting them in binary form, and comparing them. The property he mentions catches the eye and Bouton wrote as if he was examining Table 3, which he had built, but without telling us how.

3.2 The number of safe locations .

A last point needed a proof, that is that **all and only** the positions with sums per column all equal to zero, using the sum mod.2, were represented in the empirical construction of the type, which I gave in Paragraph 3.1.

One answer to this question is to calculate the total number of safe positions both theoretically and empirically. The results must be the same. Let's see how we can theoretically calculate, in the simplest way possible, the total number of safe positions. For example, let's verify that **7 is the safe position number for piles not exceeding 7 coins.** If one does this with a maximum of 7 coins per pile, he will have to fill a 3 x 3 square matrix with zeros and ones.

The number is limited because a column can give a sum mod 2 equal to zero only if it takes one of the 4 forms.

0	1	0	1
0	1	1	0
0	0	1	1

Table 5.

The 4 possible columns whose Mod2 sum (bitwise, i.e. without carryovers) of the elements is 0.

Each of the four columns can combine with all the columns of the quartet. The binary forms of numbers up to 7 only require three columns, so we can have up to 4^3 combinations, which is a total of 64 positions. However, we can subtract as less interesting the positions of the form (0, n, n,), seven of them, and their permutations (we can put the 0 in any of the three places), for a total of 7 x 3 = 21 positions (which have been dealt with separately). This operation brings us to 43 combinations. The position containing "only zeros" is then subtracted, as it is even less interesting. We are now at 42 positions. But looking at the rows, (i.e. the number of coins in each pile), we see that each position has six equivalent permutations (i.e., numbering the **rows** as 1,2,3, we have the permutations 123, 132, 213, 231, 312, 321, all equivalent for the Nim game). So we have 42/6 intrinsically different positions, that is a total of 7, which coincides with Table 1. **The different safe positions that hold up to 7 coins are the seven positions labeled a, b, c, d, e, f, g, plus seven form positions (0, n, n) or equivalent.**

Similarly, a table built empirically by Bouton, with a maximum of 15 coins per pile, yields **35 safe positions ,** and a calculation along the lines we followed above gives the same number.

Making the calculation is easy if we consider the tables for the positions, in which all the piles have at most 2^n-1 coins. For example, we can calculate safe locations for up to 15 objects (= 2^4-1) across **three piles** . So the total number of combinations is 4^4 = 256. We need to subtract the 15 x 3 (= 45) combinations, which include an empty pile. This leaves us with 211 combinations. We subtract the only "all zero" position and get 210. But each position (or line) appears 6 times (by permuting the lines), which means that the total number of different safe positions for n = 16 is 210/6 = 35 , as reported by Bouton in his article.
Again, we need to add to them the 45 positions (3x15) where two columns are equal.

1 2 3	2 4 6	3 4 7	4 8 12
1 4 5	2 5 7	3 5 6	4 9 13
1 6 7	2 8 10	3 8 11	4 10 14
1 8 9	2 9 11	3 9 10	4 11 15
1 10 11	2 12 14	3 12 15	
1 12 13	2 13 15	3 13 14	
1 14 15			

5 8 13	6 8 14	7 8 15
5 9 12	6 9 15	7 9 14
5 10 15	6 10 12	7 10 13
5 11 14	6 11 13	7 11 12

Table 6

The 15 safe positions "Nim" for three piles with a maximum of 15 coins (from Bouton's card). The 15 (x3) module positions (0, n, n,) and the position (0,0,0) are not included.

Thus, there are no defectors, which have zero sum for each column and are not safe positions.

At that point Bouton had nothing but a conjecture. He had to turn it into a theorem or a set of theorems.

3.3 Bouton's theorems.

Bouton immediately after presenting "a safe position" (Section 2) gives two theorems, **the first,** that *if A leaves a safe position on the table, then B cannot perform any remove that leaves a safe position on the table* (which would put A in difficulty). This is almost obvious, based on **Lemma 3.1.c,** because two elements of a safe position uniquely determine the third (assuming there are no empty piles).

Hence, since B can move coins from only one pile, he is forced to abandon a safe position, whatever remove he makes, because two safe positions cannot have the same number of coins in two different piles.

The **second theorem** is that *regardless of changes made to B in one column of a safe position left by A, A can always remove coins from one of the remaining two columns to create a safe position.* From what has been said above, it is obvious that one of the **two remaining piles has** to be operated (i.e. remove coins from) . Here Bouton makes a "limited proof", which considers only three piles, and, furthermore, assumes that the insecure position from which part A was left by B, which in turn started from a safe position, left by A. *Both restrictions, as we shall see, are not necessary….* But he did it first. I will give a more general proof of the second theorem than Bouton's.

He further states that the same rules apply to a game with more than three piles: " *In this case a safe combination is a set of numbers such that, when written in the binary scale and arranged with the units in the same vertical column, the sum of each column is even, i.e. 0 mod. 2.* By adding column by column (**which is called a "bit by bit" sum, basically a binary sum without carryovers) mod 2,** we have 0 0 0 0. **The sum "bit by bit mod2" will hereinafter be called NimSum [.., .., ...]** and the numbers on which the NimSum operates will be contained in square brackets. "Bitwise mod2" or "no carry" seem to me pleonasms: "binary sum mod2" excludes carry and remains bit by bit.

> I recall that in mod.2 operations the numbers are represented by the remainder of their division by 2 (i.e. either 0 or 1), and the results of all operations are represented by their remainders of division by 2. For example 3 (decimal) + 4 (decimal) is represented by 1 + 0 = 1, which is also the remainder of the division of the decimal result 7 divided by 2. In NimSum the sum of two

numbers, the only operation that matters to us) is performed by writing the two numbers in binary form, for example 3 + 5 = 011 + 101, by placing them in a column and then adding the columns
011
101
- - -
110

But, returning to the decimal form, we have that 3 + 5 = 6.

It is worth noting that the NimSum operation has **commutative** and **associative properties** and also, as shown in the addition table, **NimSum [x, x,] = 0,** which can be proved by first setting x = 0 and then setting x = 1 . However, the result can be applied to any two **equal** numbers regardless of the number of binary digits 0 and 1). For example, suppose we want NimSum [13,13], which, in binary notation, is written

1 1 0 1 +
1 1 0 1

The result (of the binary sum Mod.2 - automatically without carry) is NimSum [13,13] = **0 0 0 0** or simply zero. On the other hand, the sum of two different numbers cannot be zero and is positive (we have no negative numbers in our system.)

The example shows that if we put into columns two equal binary expressions, we always have two pairs of 1's or two pairs of 0's in each column and their NimSum is always 0.

3.4 Variants

In his Section 5, Bouton briefly considers the case of n piles. The "safe positions" have the same properties for all pile numbers from 2 onwards, and "the proof by induction [*of the theorem that the player who first sets a safe combination can do so on each subsequent remove and will win*] is so direct that it seems superfluous to give it. ”Many thanks.

In Section 6, Bouton notes that the Nim game can also be played in a version where the winner is the one who forces the other player to take the last coin (the so-called “misère” version).

I provide here my version of the strategy to win the "Misère" version, which differs somewhat from Bouton's approach.

Basically the strategy changes from the “normal” game to the “misère” one only at the end of the game, when a player (for example A) reaches the position **(1, 2, 3)** .

At that point **B** can reduce 123 to six positions: **12; 121; 122; 113; 13 or 23** .

In the “normal case”, the final winning position is (0, 1, 1) . In four cases it is immediate to see how A can reach position 011. In two cases A can easily reach (0, 2, 2) which we know is a safe position for "normal" play.

In the "Misère" game, the final winning position is (1, 1, 1) or (1, 0, 0) and we can see how A can arrive at 111 or (better) 001, from any of the six given positions (12; 121; 122; 113; 13 or 23.) Player A can reduce **12** to 1; **121** to 111, **122** to 22, **113** to 111, **13** to 1, **23** to 22.

Note that **22** forces B to 12, or 02, and in both cases, A can take the last coin ("normal" play) or force B to take the last coin ("misère").

3.5 The "standard demonstration"

As announced, I will not follow Bouton's approach to demonstrate the winning strategy, because, once we introduce the NimSum concept with its immediate properties, the proof is simple and more general than Bouton's.

While Bouton starts from a "safe position" of a "3-piles game", which appears without explanation, the standard explanation (which appears for example in Wikipedia) rightly does not mention any "safe position" at all and is not based on the fact that we are dealing with three piles. The standard proof starts from an arbitrary position, regardless of how we got there, and calculates the NimSum **S** of that position. Then it is shown that, if S = 0, each variation in a single pile leads to a position in which the sum S is not zero, while if S> 0 (cannot be less than zero) it can be reduced to S = 0 by operating on a single pile.

The proof is simple. To be concrete, suppose we have three piles, whose **number of coins** in an initial position is x1, x2, x3. After writing the coin numbers in binary form, their column-by-column sum (mod2), which we called NimSum [....], gives a binary number, S. In other words, **S = NimSum [x1, x2, x3]** .

Let there be no doubt. The reader should not be surprised if in binary form **NimSum [101, 110] = 11** , which in decimal notation becomes **5 + 6 = 3.**

After the player who is faced with a situation **with any S** performs its remove, the new position is T = [y1, y2, y3], with new values for the number of coins in the piles. In truth the rules allow to act only on one pile (which we can always permute in third place), which changes its value x3 in a new y3 and y3 <x3 (in decimal or binary form) because the coins can only be removed . In all other piles the numbers are unchanged (x1 = y1, x2 = y2).

We have: T = NimSum [0, T], but the NimSum rules tell us that NimSum [x, x] = 0, or, in our case, NimSum [S, S] = 0. So we can replace 0 with NimSum [S, S].

Therefore T = NimSum [S, S, T], and, by the associative property, T = NimSum [S, NimSum [S, T]], where
NimSum [S, T] = NmSum [NimSum [x1, x2, x3] + NimSum [y1, y2, y3]].

We can rearrange T like

T = NimSum [S, NimSum [x1, y1], NimSum [x2, y2], NimSum [x3, y3]].

But, thanks to the fact that x1 = y1 and x2 = y2, we finally have

T = NimSum [S, NimSum [x3, y3]] (the only sum that differs from 0).

We now have two cases:

I) The initial position was S = 0, which means that after a remove the resulting position cannot be T = 0,

II) The starting position was S> 0, which means that the final position can become (with proper remove) T = 0.

> The whole theorem guarantees that if a player (A) sets a position with S = 0, the opponent (B), due to (I), cannot transform the position to another position with T = 0. Player A, on the other hand, thanks to (II), can always pass from a position T> 0, created by B, to a position with S '= 0, and **fewer coins** . The process will continue (T unsafe, S 'safe; T' unsafe, S '' safe ... with **fewer and fewer coins** , until there are no more coins, and A has won, because to build the position (0,0, 0), with $S^{(2k)} = 0$ means, "***to take the last coin*** ", or the last coins. We wrote S^{2k}, because we get there with an even number of moves. Therefore, starting from a safe position and playing well, the second player arrives at the safe position par excellence (0,0,0). As we have shown, the mod.2 sum and all the other approaches are only more or less sophisticated methods to characterize safe positions and show us how to get there. *The comment I just made is almost obvious, but I thought it best to write it explicitly, because I have noticed that its frequent omission in many explanations of the game's strategy leaves many players with a recipe in hand, without understanding why starting from a safe position, the first player, while playing correctly, must inevitably lose.*

While (I) is immediate, (II) requires some attention. Let's see it in detail.

If S ≠ 0, it, written as a single binary number, will have a leftmost 1 in position D (from right). The fact that there is a 1 in the binary representation of S expresses the fact that the S of the position is not equal to 0. We must therefore look for the pile xK which has a number of coins with a number 1 in that position D, let xK (D), in binary notation. It does not need to be the leftmost bit of xK. There must be at least one of these piles, otherwise the S (D) bit, the sum of the binary digits in column D, would be zero.

So let's put **yK = NimSum [S, xK],** which is interpreted as a binary number of coins.

The NimSum of bit xK (D) with S (D) will be 0, being the NimSum [1,1]. The first 1 comes from S (D), the second from xK (D)

We claim that yK <xK (number of coins), i.e. the pile of K has decreased. In fact, all the bits to the left of D remain unchanged by removing from xK to yK, while the bit D, adding it to 1, will decrease from 1 to 0, thus decreasing the value of yK by $-2^{D.}$

Changes occurring to the right of D will amount to a maximum of $+ 2^{D}-1$. Consider, for example, the binary difference of 1000-0001, (i.e. 8-1), which yields 7 (111).

It may be useful to remember that the binary formulation of a number indicates only which powers of 2 appear in the sum that reproduces the number. For example, suppose a Zero in position D from the right indicates that the power 2^{D} has been **eliminated** . Now suppose that in all positions following D to the right, up to the end of the number, there is a 1, which means that all decreasing powers from $2^{(D-1)}$ to D^{0} are present . Then the number is given by the sum of all powers from 0 to D-1, ie by the sum $1 + 2 + 2^{2} .. + = 2^{D}-1$, which is less than the 2^{D} power that we have eliminated. We must therefore conclude that in any case the number decreases. Although higher (uninvolved) powers exist in xK, the number with 2^{D} missing is less than the number where 2^{D} appears .

The bits to the right of D can be arranged in such a way as to create a situation S = 0 (those with D greater obviously do not matter)

In other words, the player can **remove xK –yK coins** from the K pile (since xK- (xK- yK) = yK). The result is that the new T will be equal to zero:

T = NimSum [S, xK, yK] =NimSum [(S, xK), NimSum [S, xK]] = 0 , cdd. (The two addends are the same.)

This means that with a single remove, you can move the game from any insecure position to a safe position.

An example will (I hope) clarify the situation and at the same time show what happens if there are multiple piles whose binary representation has a 1 where the highest bit of S is (i.e. in the position we named D). I have not seen this particular case dealt with elsewhere, but it is not impossible to find, at least as a starting position. Suppose we have the three piles **(6, 3, 7), that is (110, 011, 111).** If both players play well the (6,3,7) position is only possible as a randomly chosen starting position, which we assume, while it is an impossible position in the course of a correct game *(Why? Hint: it is impossible to get (6,3,7) from a secure location with multiple coins)* . The NimSum of the three piles is 010. As you can see, the first bit to the left of S is at D = 2, and all three piles have 1 as the 2nd bit.

1 1 0
0 1 1
1 1 1
S = 0 1 0

Now, S can become 0 if we put 0 in place of 1 in second place (from the right) in any of the binary numbers that represent each pile. This can be achieved by NimSum [S, xK] (in this case K is 1, or 2, or 3, meaning any of the three piles). We therefore have three possibilities: putting 0 in the middle of the top row (and remember that **4, 3, 7** = 3, 4, 7); putting 0 in the middle line (**6, 1, 7** = 1,6,7); putting 0 in the bottom row (**6, 3, 5** = 3, 5, 6). All three positions we get are safe.

The results were achieved by inspection, but we should demonstrate that the same result can be achieved by following the above rules. We consider yK = NimSum [S, xK].

- If we select **K = 1** , we have y1 = NimSum [010, 110] = 100, corresponding to **y1 = decimal 4;**
- for **K = 2** we have y2 = NimSum [010, 011] = 001, corresponding to **y2 = decimal 1** ;
- for **K = 3** , we have NimSum y3 = [010, 111] = 101, corresponding to **y3 = decimal 5.**

By subtracting from x1 the difference x1-y1 = 6-4 = 2 from the first pile we obtain the three piles **4, 3, 7;** subtracting 3-1 = 2 from the second pile we have **6,1,7;** subtracting 7-5 = 2 from the third pile, we have **6,3,5** . As above, they are all safe positions.

> In addition to practicing on the way to reach a safe position from an unsafe position, we have discovered a class of positions, which cannot be reached in the course of a fair game. However, they can present themselves as a starting position, especially if the starting position is randomly selected, to ensure fair play.

As we can see, *at no time does the standard proof depend on the number of piles,* which can be as many as we want. A player who lands on a position S = 0 (which is safe) always wins, and the rules for defining and creating (if necessary) a safe position also remain the same. Of course, both players have to play a perfect match.

So what is the strategy for winning?

The Nim game is truly a curious game. To make it an honest game, you need to draw lots for the numbers of coins for each pile, and to decide who is the first player. At this point, however, we already know who will win, and, **between two good players, it is useless to play the game.**

starting position	first to play	WINNER
SAFE	A	B
	B	A
NOT SAFE	A	A
	B	B

We thus have the result, common to all games for which it is possible (mathematically) to prove that there is a winning strategy, that **they, from mathematical games become games of chance,** because the victory is determined by the initial conditions, which, in order to have a fair game, must be drawn by lot. Since the draw by lot determines the outcome, it becomes useless to play the game. ***Theorem: Mathematical games destroy themselves.***

Conclusion

At the end of his relatively short life (he died at the age of 53) Charles Leonard Bouton had to suffer from health problems, family problems, personal pain.

Considering the main area of his research in his early years, I think that the essay on the game of Nim (a name, which Bouton himself coined, probably from the German word Nimm, which means " take!" - but without specidying the reasons for his choice) was not taken too seriously by its author. Rather, he must have regarded it only as mathematical amusement.

However, while no fundamental contributions of his are mentioned in the annals of differential equations, **today his article** ***" Nim, a game with a complete mathematical theory "" (1901) on Annals of Mathematics Vol. 3, n. 1/4, 1901 - 1902*** **is believed to have laid the foundations of the combinatorial game theory**, an entire field of mathematical research. Good for him!

As a concluding remark I add that the interest in the game was revived by a famous film: " L'Année dernière à Marienbad", a 1961 Italian-French Left Bank film, directed by Alain Resnais, from a screenplay by Alain Robbe-Grillet There are three characters and one, named M, continually defeats the main character (?), X, and says, "I can lose, but I always win."

Wikipedia comments [I put my comments in square brackets]: The film is famous for its enigmatic narrative structure, in which time and space are fluid, with no certainty about what is happening to the characters, what they are remembering or what they are imagining. Its dreamlike nature captivated [the critics] and baffled viewers; many hailed the work as an avant-garde masterpiece , although others [the majority,] found it incomprehensible. ... *Marienbad* received an entry in *The Fifty Worst Films of All Time* , ...

Fig. 1: Neues Schloss Schleißheim

This is NOT the hotel referenced in the movie "Last Year in Marienbad" (but close to it).

NOTE

(1) Charles L. Bouton (Saint Louis (MO) 1869 - Cambridge (MA) 1922) was a professor at Harward University, where he was remembered as an outstanding teacher, and editor / co-editor of two mathematical journals. He had studied for two years in Leipzig (with a Parker scholarship), where he had been one of the last students of the Norwegian mathematician Sophus Lie (died February 18, 1899). He received his PhD in 1898, with the thesis " *Invariants of the General Linear Differential Equations and their Relation to the Theory of Continuous Groups,"* Supervisor: Sophus Marius Lie.

(https://www.ams.org/journals/bull/1922-28-03/S0002-9904-1922-03508-2/S0002-9904-1922-03508-2.pdf)