An introduction to the HESSIAN

2nd Edition

Photograph of Ludwig Otto Hesse (1811-1874), circa 1860

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I.

As the students of the first two years of mathematical analysis well know, the Wronskian, Jacobian and Hessian are the names of three determinants and matrixes, which were invented in the nineteenth century, to make life easy to the mathematicians and to provide nightmares to the students of mathematics.

II.

I don't know how the Hessian came to the mind of Ludwig Otto Hesse (1811-1874), a quiet professor father of nine children, who was born in Koenigsberg (I think that Koenigsberg gave birth to a disproportionate number of famous men). It is possible that he was studying the problem of finding maxima, minima and other anomalous points on a bi-dimensional surface. (An alternative hypothesis will be presented in section X.)

While pursuing such study, *in one variable,* one first looks for the points where the first derivative is zero *(if it exists at all),* and then examines the second derivative at each of those points, to find out its "quality", whether it is a maximum, a minimum, or an inflection point. The variety of anomalies on a bi-dimensional surface is larger than for a one dimensional line, and one needs to employ more powerful mathematical instruments. Still, also in two dimensions one starts by looking for points where the two first partial derivatives, with respect to x and with respect to y respectively, exist and are both zero. A point on the surface where both first partial derivatives are zero is called a "**critical point**". One must then investigate the nature of the critical points, whether they are maxima, minima, saddle points or anything else.

III.

To introduce the concept of "Hessian" I will content myself with a heuristic approach through the Taylor expansion in two variables.

In one variable, we know that

$$f(a+\Delta x) = f(a) + f'(a)\,\Delta x + \frac{1}{2} f''(a)\,(\Delta x)^2 \ldots$$

For our discussion of the "Hessian" we won't need more that the second degree, and therefore we will stop there. Of course the function we investigate may include higher degree powers of x and y, or even transcendent functions, but, as we do with functions of one variable, investigating the second derivative terms will be enough, in most situations.

For completeness I will show here a heuristic, non-rigorous method to expand a function f(x) in Taylor series.

The polynomial (a constant)

$p_0(x) = f(a)$

has the same value as f(a), at the point a. We will now try to improve the polynomial in such a way that also its first derivative be equal to the first derivative of the function, still at the point x=a.

To obtain this result we add a first degree term B(x – a) and determine the coefficient B by imposing that the first derivatives are equal:

$p_1(x) = f(a) + B\,(x - a)$

From which:

$p'_1(a) = B = f'(a)$.

We now want that also the second derivatives are equal at x=a, and we must add a second degree term:

$p_2(x) = f(a) + f'(x - a) + C\,f''(x - a)^2$

Which entails,

$p''_2(a) = 2C = f''(a)$

Our polynomial is now

$p_2(x) = f(a) + f'(a)(x - a) + \frac{1}{2} f''(a)(x - a)^2$

We can continue adding terms of increasing degree. First we will obtain a term of the third degree, with

$p'''_3(a) = 3 \cdot 2D = f'''(a)$

And then all the others, until infinity. We can also set $(x - a) = \Delta x$, from which

x = $a + \Delta x$ and in the end :

$$f(a + \Delta x) = f(a) + f'(a)\,\Delta x + \frac{1}{2} f''(a)\,(\,\Delta x)^2 \ldots$$

The strictness of such derivations, both of the Maclaurin series and Taylor series is zero, but the result is correct, and the method gives at least an idea , which possibly guided the discoverers to their developments. It should be noticed that in the procedure we have adopted, almost new or thoroughly new ideas were introduced:

1) That infinite polynomials could exist and make sense;

2) That an infinite polynomial, which at a given point had all derivatives equal to those of a given function, could approximate it even at points very far from the starting point, which is a non-intuitive, non-trivial hypothesis.

Ad abundantiam I include a figure which shows how the terms of increasing degree progressively added to the polynomial make it to approximate better and better the original function (here f(x) = sin(x)) closer and closer

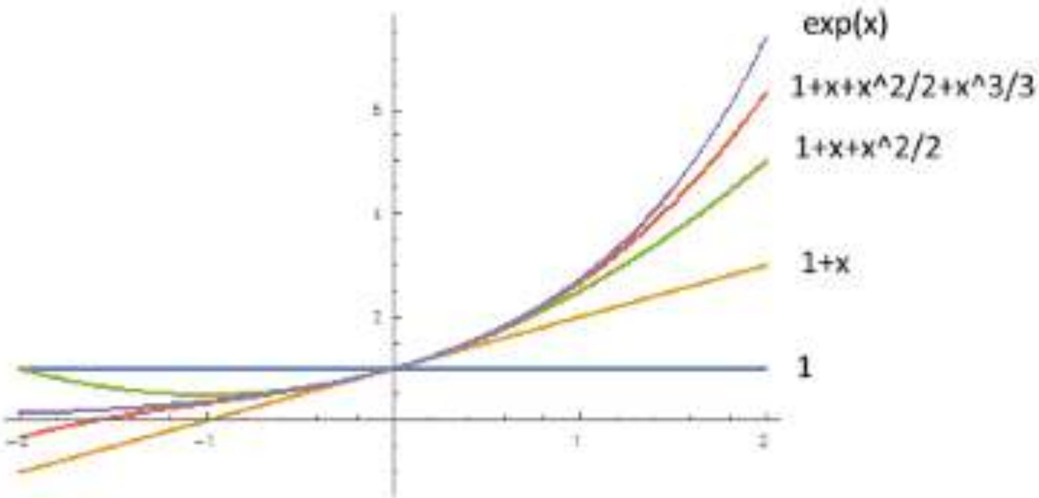

Fig.0, made with Wolfram Mathematica

In two variables the expansion around the point of coordinates (a, b) is a bit more complicated.

Proceeding as in the case of one variable, we first of all must have $f(a,b) =$ *p0x,y=costant*

We now add a first degree term and impose that the first derivatives of the new polynomial and of the function are equal in the point (a, b). We should just remember that the first degree terms are two, B(x-a) e C(y-b). We must impose that

$f(x,y) = A + B(x-a) + C(y-b)$

Taking the derivatives and evaluating them in the point (a, b), we have:

B = $\left(\frac{\partial f}{\partial x}\right)_{a,b}$ and C = $\left(\frac{\partial f}{\partial y}\right)_{a,b}$

Which we will soon forget, because, as in the one variable case, we will find the "critical points" of the function precisely by imposing that its first derivatives are equal to 0.

We must now calculate the second order derivatives of

$D\ (x-a)^2 + E\ (x-a)(x-b) + F\ (y-b)^2 \ldots$

In the point x = a, y = b.

Per prima cosa si calcola la derivata seconda rispetto a x del primo termine, che ci dà 2D, da cui:

$$D = \frac{1}{2}\left(\frac{\partial}{\partial x}\frac{\partial f}{\partial x}\right)_{a,b}$$

The second derivative of $F\ (y-b)^2$ with respect to y is 2F. Therefore,

$$F = \frac{1}{2}\left(\frac{\partial}{\partial y}\frac{\partial f}{\partial y}\right)_{a,b}$$

The only remaining term is $E\ (x-a)(x-b)$, which is to be calculated in two steps, whose order, luckily, is irrelevant.

The first derivative if the term with respect to x gives

$$\frac{\partial (x-a)(y-b)}{\partial x} = E\ (y-b)$$

While the first derivative of the resulting $E\ (y-b)$with respect to y gives E.

We set this result to be equal to the second derivative of the function with respect to x and y, calculated in the point (a, b):

$$\left(\frac{\partial}{\partial x}\frac{\partial f}{\partial y}\right)_{a,b} = E$$

Following this procedure, some find it difficult to understand why the various derivatives of the function must be calculated in the point (a, b). But this was precisely the starting program: to create a polynomal , even of infinite degree, such that all its derivatives are equal to the derivative of the function in the given point (a, b) . In fact, if the calculate, for example, the second derivatives of the polynomial, one sees that they are equal to the second derivatives of the function in the point (a,b), because the second derivatives of all higher degree terms of the polynomial contain terms (x-a) or (y-b), which become 0 when x=a and/or y =b.

We thus have the expansion:

$f(x,y) = f(a,b) + \left(\frac{\partial f}{\partial x}\right)_{a,b}(x-a) + \left(\frac{\partial f}{\partial y}\right)_{a,b}(y-b) + \frac{1}{2}\left(\frac{\partial}{\partial x}\frac{\partial f}{\partial x}\right)_{a,b}\ (x -$

a)2+∂∂x∂f∂ya,bx−a(y−b)+12 ∂∂y∂f∂ya,b (y−b)2 …

There remains only one more step to be done: we can remark that we can write, for fairly small increments: x - $a = \Delta x$, *that is* $x = a + \Delta x$, and similarly y - $b = \Delta y$, *that is* $y = b + \Delta y$.

Now, bringing *f(a, b)* to the left hand side and having set equal to zero both first derivatives to find, as in the one-variable case, the "critical points", our expansion around a critical point of coordinates (a, b) will be:

$$(1)\quad \Delta f = f(a + \Delta x, b + \Delta y) - f(a,b)$$
$$= \quad +\frac{1}{2}\left(\frac{\partial}{\partial x}\frac{\partial f}{\partial x}\right)_{a,b} \Delta x^2 + \left(\frac{\partial}{\partial x}\frac{\partial f}{\partial y}\right)_{a,b} \Delta x\, \Delta y + \frac{1}{2}\left(\frac{\partial}{\partial y}\frac{\partial f}{\partial y}\right)_{a,b} \Delta\, y^2$$

All derivatives are calculated at x =a, y = b, and therefore are just ordinary numbers.

And we stop here as promised, and do not include in our analysis the higher order terms. In most cases they are not necessary.

IV.

As we said, if we expand our function near a "critical point", the terms containing the first derivatives identify the critical point by being put equal to zero.

For example, for the function $z = x^2 - y^2$, we must put the first derivatives equal to zero, thereby obtaining 2x =0 and 2y = 0, that is, that the point (x =a = 0, y = b = 0), the origin, is the critical point.

We can now consider f(x,y) as the equation of a surface in two dimensions, and examining the properties of the right hand side we should be able to understand the shape of the surface in the neighborhood of the critical point. Right, but how to do it?

V. Quadratic Forms and associated symmetric matrixes.

In general an expression of the type

$$Q(u,v) = Au^2 + 2C\, u\, v + Bv^2$$

(which can be extended to n variables, provided all terms are of second degree) is called a "quadratic form" and has many theoretical and practical applications. In our case it is associated to a symmetric matrix, as one can see by developing the product

$$(u\,,v)\begin{pmatrix} A & C \\ C & B \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix} = (u\,,v)\begin{pmatrix} Au + Cv \\ Cu + Bv \end{pmatrix} = Au^2 + 2C\,u\,v + Bv^2$$

It must have occurred to Hesse that the right hand side of (1) is the quadratic form resulting from the product of a symmetric (HESSIAN) MATRIX by a bi-dimensional vector $\begin{pmatrix} \Delta x \\ \Delta y \end{pmatrix}$ and by its transposed $(\Delta x\,, \Delta y)$ (all multiplied by ½, a factor which we will henceforth omit).
With "obvious" notation for the second derivatives (f_{xx}, f_{yy}, f_{xy}), we obtain:

$$(\Delta x\,, \Delta y)\begin{pmatrix} f_{xx} & f_{xy} \\ f_{xy} & f_{yy} \end{pmatrix}\begin{pmatrix} \Delta x \\ \Delta y \end{pmatrix} = f_{xx}\,\Delta x^2 + 2f_{xy}\Delta x\,\Delta y + f_{yy}\Delta y^2$$

Some confusion inevitably arises when one talks about "Hessian". Generally it means the Hessian matrix $((H(f(x,y))$, but many beginners, not without some reason, think that one is talking about the Hessian determinant,

$$Det(H(f(x,y)) = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{xy} & f_{yy} \end{vmatrix} = f_{xx}f_{yy} - f_{yy}{}^2$$

which so far has not yet come into play. Its turn will come, though.

Let's leave for a moment the determinant aside. We have

$$(1b) \quad \Delta f = f(\mathrm{a} + \Delta x, \mathrm{y} + \Delta y) - f(a,b)$$
$$= +\frac{1}{2}\left(f_{xx}\ \Delta x^2 + 2\, f_{xy}\Delta x\,\Delta y + f_{yy}\ \Delta y^2\right)$$

which means that **if** the "quadratic form", which we have on right-hand side is always positive , in whatever direction we move away from the critical point of coordinates a and b, that is whatever are the components Δx, Δy, we have that the difference between the new value $f(\mathrm{a} + \Delta x, \mathrm{y} + \Delta y)$ and the function at the critical point is positive. In other words, $f(\mathrm{a} + \Delta x, \mathrm{y} + \Delta y)$ is always "above" $f(a, b)$, which is therefore a minimum.

On the other hand if the quadratic form on the right is always negative, in whatever direction we move from the critical point, we have that the difference between the new value $f(\mathrm{a} + \Delta x, \mathrm{y} + \Delta y)$ and the function at the critical point is negative. Then $f(\mathrm{a} + \Delta x, \mathrm{y} + \Delta y)$ is always "below" $f(a, b)$, which is therefore a maximum.

By the same reasoning, if in some direction the quadratic form is negative and in other directions is positive, we have a "saddle point".

Finally, if the quadratic form is equal to zero, we cannot come to any conclusion and we have to deepen our analysis, a task we will not enter in detail.

One question which might arise is how to deduce the quadratic form from the symmetric matrix and vice versa. As long as we are working in two dimensions, the answers can be given almost by simple inspection. If we move to a higher dimension, the vice versa is still easy : one multiplies the matrix on the right by the vector column $\begin{pmatrix} x \\ y \\ z \\ u \end{pmatrix}$ and on the left by the transposed (row) vector $(x,y,z,u\ldots)$

Going from the Quadratic form to the associated matrix is as easy. One must only remember that the quadratic form is given by

$$\mathrm{Q}(x_1, x_2, x_3, x_4 \ldots) = \sum_1^n a_{rs}\, x_r x_s$$

and reorder the coefficients. (It helps to remember that the matrix is simpler to use if it is symmetric).

Example:

Suppose we want to find the matrix associated to the quadratic form

$$Q(x,y) = x^2 + 2\,xy - 3yx + 3y^2$$

Re-baptising $x = x_1, y = x_2$, we can write

$Q(x_1, x_2) = 1\,x_1\,x_1 + 2\,x_1x_2 - 3x_2x_1 + 3\,x_2x_2$

From which: $a_{11} = 1, a_{12} = 2, a_{21} = -3, a_{22} = 3$

The matrix is thus

$$\begin{pmatrix} 1 & 2 \\ -3 & 3 \end{pmatrix},$$

which is not symmetric. However, -3yx + 2xy = - xy, and one can impartially divide -1= -0.5 - 0.5, thus obtaining

$$\begin{pmatrix} 1 & -0.5 \\ -0.5 & 3 \end{pmatrix}$$

Which gives us back the quadratic form

$$Q(x,y) = x^2 - xy + 3y^2 .$$

In fact one can generalize the above example by giving the prescription that

$$a_{ij} = \frac{1}{2}(a_{ij} + a_{ji})$$

If $a_{ij} = a_{ji}$ from the beginning, "*tant mieux*", as a Frenchman would say.

VI.

Thus the question is now, how do we decide the behavior of the term on the right of 1b? One method would be by "brute force", that is tracing a (reasonably small circle, *not to include other critical points)* centered on the critical point under exam, calculating the value of $f\,(x, y)$ for as many points of the circle as possible and comparing the values so obtained with the value of the function at the critical point. We would then know whether we are dealing with a maximum, a minimum or a saddle point (by the way, we would also solve the uncertain situation arising when the quadratic form is equal to 0).

The ideal result would be that of demonstrating that a quadratic form can easily be shown to have always a positive, or negative, value, whatever are the directions of the vectors ($\Delta x\,\vec{\imath}$, $\Delta y\,\vec{\jmath}$) from the critical point, without calculating the values of the quadratic form point by point.

VI.

Such ideal objective can be achieved and a little knowledge of the theory of eigenvalues will help us to understand the trick. I shall deal for example sake only with 2 x 2 matrices. One can demonstrate that a symmetric 2 x 2 matrix always has real eigenvalues, which are found remembering the foremost meaning of **eigenvalues and eigenvectors** of a matrix, such as:

$$\begin{pmatrix} a & c \\ c & b \end{pmatrix}$$

If we multiply a matrix by a vector of components $\binom{x}{y}$, *in general we find a different vector. However, there are cases in which we find the same vector, at least in direction,* albeit shortened or lengthened by a factor traditionally called λ, the eigenvalue . Such a vector is called eigenvector. To find it, we must therefore solve the simple algebraic system

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}\binom{x}{y} = \lambda\binom{x}{y}$$

Which, for a symmetric matrix, becomes:

$$\begin{cases} ax + cy = \lambda x \\ cx + by = \lambda y \end{cases}$$

Thus (x, y) becomes (λx, λy), that is, the same vector lengthened – or shortened – by a factor λ.

The first grade algebraic system readily becomes:

(2)

$$\begin{cases}(a-\lambda)x + cy = 0 \\ cx + (b-\lambda)y = 0\end{cases}$$

We now have a homogeneous system, and the determinant of the coefficients of x, y, must be 0, otherwise we will have only one solution. Such a solution, being unique, can only be the useless (x = 0, y = 0), which indeed satisfies the system. The determinant of the coefficients results in the second degree equation:

$$\lambda^2 - (a+b)\lambda + ab - c^2 = 0$$

Here we see that the discriminant of the equation

$$[-(a+b)]^2 - 4a\,b + 4c^2 = (a-b)^2 + 4c^2$$

is always positive, which guarantees that our equation has real roots. This is due to the symmetry of the matrix. The two real roots we will obtain make the determinant equal to zero, which is what we want, in order to have non-trivial solutions.

Of the above equation it will be useful to remember that:

$$\lambda_1\lambda_2 = ab - c^2$$
$$\lambda_1 + \lambda_{2=} +(a+b)$$

VII.

Having found the two real eigenvalues (λ_1, λ_2), we can substitute them in the (2), thus obtaining two eigenvectors ($\vec{v}_1$, $\vec{v}_2$) which can be normalized by dividing each of them by $\sqrt{\vec{v}_i \cdot \vec{v}_i}$, a product which some call Norm, while others call Norm its square. It is just a matter of names.

Although, once the eigenvalues are known, it is not difficult to calculate the two eigenvectors, I have seen many students in difficulty in front of this problem. Yet, starting from any of the two equations (it does not matter which one, because, as the determinant of the coefficient, once the eigenvalues are inserted, is zero, they are equivalent) we have that

$$(a - \lambda)\mathrm{x} + cy = 0$$

The equation is satisfied, for example, by taking

$$\frac{x}{y} = -\left(\frac{c}{a - \lambda}\right)$$

It then follows:

$$\vec{v}_1 = \begin{pmatrix} c \\ \lambda_1 - a \end{pmatrix} \text{ and } \vec{v}_2 = \begin{pmatrix} c \\ \lambda_2 - a \end{pmatrix}$$

Whose internal product is given by

$$\mathrm{c}^2 + (\lambda_1 - a)(\lambda_2 - a) = c^2 + \lambda_1\lambda_2 - a(\lambda_1 + \lambda_2) + a^2$$

Using the formulas for $\lambda_1\lambda_2$ and $(\lambda_1 + \lambda_2)$ mentioned above::

$$= c^2 + (ab - c^2) - a\,(a + b) + a^2 = 0$$

Thus the two eigenvectors can be taken as a new orthonormal basis of the space in 2 dimensions.

Stop! The lazy but fussy student might say. The orthogonality is OK, but how can we be sure of the normality? The answer is that we ourselves fix the normality by dividing the vector by its norm, or its square root, according to the convention we decide to follow), in such a way that, by multiplying it by itself or – better - by its transposed, we obtain 1.

Thus, to normalize the vector $\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$ we perform the internal product: N = $(u_1\ , u_2)\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \ = \ u_1^2 + u_2^2$, we take the square root and divide the original vector by it.

We thus obtain the normalized vector

$$\begin{pmatrix} U_1 \\ U_2 \end{pmatrix} = \frac{1}{\sqrt{u_1^2 + u_2^2}}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$$

Which, multiplied by its transposed vector

$$(U_1, U_2) \ = \frac{1}{\sqrt{u_1^2+u_2^2}}(u_1\ , u_2)$$

Produces :

$$\frac{1}{\sqrt{u_1^2+u_2^2}}(u_1\ , u_2)\ \frac{1}{\sqrt{u_1^2+u_2^2}}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \ = \ \frac{u_1^2+u_2^2}{u_1^2+u_2^2} = 1$$

In other words, now we have a normalized vector. From now on we will assume in general that our eigenvectors are normalized.

The advantage is that using as a new basis the eifgenvectors

(i) *A symmetric 2 x 2 matrix M can assume a diagonal form (the two diagonal terms being the (real) eigenvalues)*. It is easy to show why.

We have that

$$\text{M}\ \vec{\boldsymbol{v}}_{\boldsymbol{1}} = \lambda_1 \vec{\boldsymbol{v}}_{\boldsymbol{1}}$$

$$\text{M}\ \vec{\boldsymbol{v}}_{\boldsymbol{2}} = \lambda_2 \vec{\boldsymbol{v}}_{\boldsymbol{2}}$$

And therefore, multipliying the first equation by $\vec{\boldsymbol{v}}_{\boldsymbol{1}}$, on the left

$$\vec{\boldsymbol{v}}_{\mathbf{1}} M\, \vec{\boldsymbol{v}}_{\mathbf{1}} = \vec{\boldsymbol{v}}_{\mathbf{1}}\, \lambda_1\, \vec{\boldsymbol{v}}_{\mathbf{1}}$$

The left hand side is $M'_{1\,1}$, the new (1,1) element of the matrix M in the new basis. On the right hand side we have

$$\vec{\boldsymbol{v}}_{\mathbf{1}}\, \lambda_1\, \vec{\boldsymbol{v}}_{\mathbf{1}} = \lambda_1 \vec{\boldsymbol{v}}_{\mathbf{1}} \vec{\boldsymbol{v}}_{\mathbf{1}}$$

because λ_1 is just a number, and can be put before or after the vector it multiplies. Besides, $\vec{v}_1$ is normalized, and therefore we have $\vec{v}_1\vec{v}_1 = 1$, and

$$\vec{\boldsymbol{v}}_{\mathbf{1}}\, \mathrm{M} \vec{\boldsymbol{v}}_{\mathbf{1}} = M'_{11} = \lambda_1$$

As the eigenvectors are orthogonal, besides being normalized, we have also:

$$\vec{\boldsymbol{v}}_{\mathbf{2}}\, \mathrm{M}\, \vec{\boldsymbol{v}}_{\mathbf{1}} = M'_{\mathbf{12}} = \vec{\boldsymbol{v}}_{\mathbf{2}}\, \lambda_1\, \vec{\boldsymbol{v}}_{\mathbf{1}} = \mathbf{0}$$

Operating likewise on

$$\mathrm{M}\, \vec{\boldsymbol{v}}_{\mathbf{2}} = \lambda_2 \vec{\boldsymbol{v}}_{\mathbf{2}}$$

with $\vec{\boldsymbol{v}}_{\mathbf{1}}, \vec{\boldsymbol{v}}_{\mathbf{2}}$ on the left, **we** finally get the diagonal form in the new basis:

$$\mathrm{M}' = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$$

If you look back with "different eyes", you see that we have performed a piecemeal transformation of a Matrix, which, putting everything together, is:

$$\begin{pmatrix} \boxed{\vec{v}_1} \\ \boxed{\vec{v}_2} \end{pmatrix} \begin{pmatrix} a & c \\ c & b \end{pmatrix} \begin{pmatrix} \boxed{\vec{v}_1} & \boxed{\vec{v}_2} \end{pmatrix}$$

where the boxes are the two-component new basis vectors, in green the vectors, in red their transposed or "dual" vectors.

It is perhaps worth mentioning that the basis vectors we imply (generally without realizing it) while writing the elements of a generic 2x2 matrix M are

$$\vec{\imath} = \begin{pmatrix}1\\0\end{pmatrix} \;;\; \vec{\jmath} = \begin{pmatrix}0\\1\end{pmatrix}$$

For example, by multiplying

$$(1,0)\begin{pmatrix}a & b\\ c & d\end{pmatrix}\begin{pmatrix}0\\1\end{pmatrix} = (1,0)\begin{pmatrix}b\\d\end{pmatrix} = b, \text{that is } M_{12}$$

But now we came to the main point of the discussion:

(ii) *All vectors in two dimensions can be expressed in terms of the two eigenvectors* since the two eigenvectors form an orthonormal basis. We just have to expand the unit vectors of the old basis, $\vec{\boldsymbol{\imath}}$ and $\vec{\boldsymbol{\jmath}}$ in terms of the new basis $\vec{v}_1$and $\vec{v}_2$, as follows:

$$\vec{\boldsymbol{\imath}} = (\vec{\boldsymbol{v}}_1 \cdot \vec{\boldsymbol{\imath}})\vec{\boldsymbol{v}}_1 + (\vec{\boldsymbol{v}}_2 \cdot \vec{\boldsymbol{\imath}})\vec{\boldsymbol{v}}_2$$
$$\vec{\boldsymbol{\jmath}} = (\vec{\boldsymbol{v}}_1 \cdot \vec{\boldsymbol{\jmath}})\vec{\boldsymbol{v}}_1 + (\vec{\boldsymbol{v}}_2 \cdot \vec{\boldsymbol{\jmath}})\vec{\boldsymbol{v}}_2$$

Thus, if we select our displacement vector as $\Delta\vec{r} = \vec{\boldsymbol{\imath}}\,\Delta x + \vec{\boldsymbol{\jmath}}\,\Delta y$, it becomes:

$$\Delta\vec{r} = [(\vec{\boldsymbol{v}}_1 \cdot \vec{\boldsymbol{\imath}})\vec{\boldsymbol{v}}_1 + (\vec{\boldsymbol{v}}_2 \cdot \vec{\boldsymbol{\imath}})\vec{\boldsymbol{v}}_2]\,\Delta x + [(\vec{\boldsymbol{v}}_1 \cdot \vec{\boldsymbol{\jmath}})\vec{\boldsymbol{v}}_1 + (\vec{\boldsymbol{v}}_2 \cdot \vec{\boldsymbol{\jmath}})\vec{\boldsymbol{v}}_2]\,\Delta y$$

or also

$$\Delta\vec{r} = [(\vec{\boldsymbol{v}}_1 \cdot \vec{\boldsymbol{\imath}})\,\Delta x + (\vec{\boldsymbol{v}}_1 \cdot \vec{\boldsymbol{\jmath}})\Delta y]\vec{\boldsymbol{v}}_1 + [(\vec{\boldsymbol{v}}_2 \cdot \vec{\boldsymbol{\imath}})\,\Delta x + (\vec{\boldsymbol{v}}_2 \cdot \vec{\boldsymbol{\jmath}})\Delta y]\vec{\boldsymbol{v}}_2 = c_1\vec{v}_1 + c_2\vec{v}_2,$$

and one can see by inspection that c_1 and c_2 are indeed the components of $\Delta\vec{r}$, along the new orthonormal basis $\vec{\boldsymbol{v}}_1, \vec{\boldsymbol{v}}_2$.

But a diagonal matrix allows us to write the quadratic form in the so-called "canonical form".

In fact, we go from the old bi-dimensional system in which:

1. the basis were the unit vectors $\vec{\boldsymbol{\imath}}, \vec{\boldsymbol{\jmath}}$,
2. the (symmetric) matrix associated to the quadratic was M, resulting in
3. …the associated quadratic form Q(M) = $ax^2 + 2bxy + y^2$

to the new system, in which

1. the basis are the unit vectors $\vec{\boldsymbol{v}}_1, \vec{\boldsymbol{v}}_2$,
2. the symmetric matrix associated to the quadratic is S′ = $\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$,
3. the associated quadratic form Q(M′) = $(c_1, c_2) \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} =$ $\lambda_1 c_1^2 + \lambda_2 c_2^2$

Thus, the associated quadratic form has assumed the **CANONICAL FORM:**

(3) $$Q(M') = \lambda_1 c_1^2 + \lambda_2 c_2^2 .$$

VIII.

In the case under study, therefore, as both c_1^2 and c_2^2 are positive numbers, the most important feature, which is called the "signature" of the quadratic form, depends on the eigenvalues of the Hessian matrix: the quadratic form associated to H (the Hessian), Q(H), will always have a positive value (condition for a minimum of the surface) if both eigenvalues will be positive . It will be always negative if both eigenvalues will be negative (condition for a maximum of the surface**). In both cases their product, the value of the determinant, will be positive. But this means that it will be sufficient to know the sign of one eigenvalue to know whether we are in a minimum or maximum condition, since the other eigenvalue must have the same sign.**

In these cases we say that the quadratic form is definite, positive or negative, with the consequences we already mentioned on *f (a, b)* being a minimum or a maximum. If one eigenvalue is zero, the quadratic becomes "semidefinite", either positive or negative, depending on the sign of the other eigenvalue.

IX.

But what if the eigenvalues have a different sign? According to 1b, page 7, we have:

$$\Delta f = \frac{1}{2}\,[Q(\Delta x, \Delta y)],$$

Which, reduced to canonical form (3), page 15, can be rewritten as:

$$\Delta f = \frac{1}{2}\,[Q(c_1,\ c_2)]$$

Where $c_1,\ c_2$ are just the translation of $\Delta x, \Delta y$ in the new coordinate system.

Therefore, without loss of generality, if the eigenvalues have a different sign, we can study the function:

$$z = 2\Delta f = |\lambda_1|U^2 - |\lambda_2|W^2$$

Where we have an irrelevant factor 2.

For z = 0, that is on the plane through the critical point, we have two straight lines through the origin of the plane (U, V), a common feature when the origin is a saddle point ; for $z\ (= 2\Delta f) < 0$ we have a hyperbola and our surface begins to shape up. For z > 0 we still have a hyperbola, but orthogonally oriented with respect to the first one.

Finally, we can study the cases U = 0 and W= 0. In the first case we have the curve z = - $|\lambda_2|W^2$, a convex parabola; in the second case we have q = $|\lambda_1|U^2$, a concave parabola. (See fig.1).

Three Examples.

A. The quadratic form z = $x^2 - y^2$. The critical point is the origin, because $\frac{\partial f}{\partial x}$ = 2x = 0 and $\frac{\partial f}{\partial y}$ = 2y = 0 from which x = y = 0. But, now, we must be smart. The quadratic is already in canonical form, and we can easily write the Hessian determinant as:

$$\begin{vmatrix} 2 & 0 \\ 0 & -2 \end{vmatrix} = -4$$

The two eigenvalue equations are

$$\begin{cases} (2-\lambda)\,x = 0 \\ -(2+\lambda)y = 0 \end{cases}$$

The determinant is $-(2-\lambda)(2+\lambda) = 0$, which gives the two roots, which we could have guessed just from writing down the determinant. It is equally simple to see that the eigenvectors are oriented along the orthogonal axes $y = 0$ and $x = 0$ The determinant is negative and therefore we are dealing with a saddle point.

How is the saddle oriented?

If we take the function z(x , y) at the value zero, that is we consider the function $z = x^2 - y^2 = 0$, we find that z splits into two straight lines through the origin $(x + y)(x - y) = 0$, or $y = x$ and $y = -x$, at 45° with respect to the axes of coordinates (in green in the diagrams in Fig.1).

We just have to see what happens along the old orthogonal axes: for example let's take y. In this case we find that along the y axis (that is for $x = 0$) the values are $z = -y^2$, a convex parabola, everywhere negative (except in the origin). Along the other axis, on the other hand, we will have a concave parabola, everywhere positive (except in the origin). We can thus reconstruct the saddle: in the blue sectors we are below the origin, in the red sectors we are above the origin. In the second diagram, the two parabolae are drawn in purple color, the two hyperbolae in red and blue.

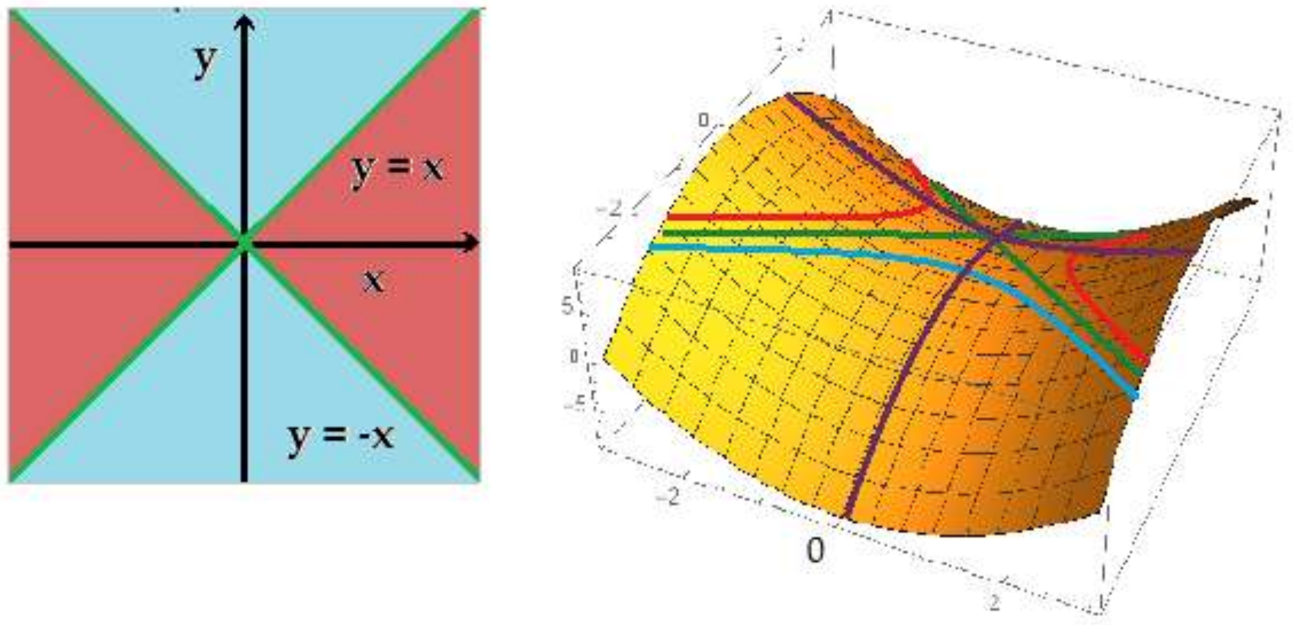

Fig.1 (partly made using the Wolfram-Mathematica language).

B. The quadratic form $z = x^2 + 2\,x\,y + y^2$**.**

Such a form is very interesting, because it appears in various problems. I think it was Poincaré, who said "Mathematics is the art of giving the same name to different things. As a consequence (but Poincaré did not say it) the same "solution method" can be applied to different "problems". But the sentence makes sense also by exchanging the words "solution methods " with the word "problems": Mathematics is therefore also the art to solve the same "problem" with different "solution methods".

The quadratic form we want to study will throw some light, I hope, also on the meaning of eigenvectors and eigenvalues. Let's examine two problems in which the form appears.

(1) First problem: small oscillations of a system consisint of two mass points connected by elastic strings.

Let's consider the oscillating system (Fig.2) , simplified to the utmost, consisting of two equal point masses, which, at rest, stand on a straight line at points situated at the same distance 1 between themselves and between themselves and two fixed extremes of the string. The elastic force constants are assumed to be equal. Let's also make the hypothesis that the vertical displacements from the horizontal line to the ponts B and C (i.e. x and y) are uniquely vertical and small with respect to to 1.

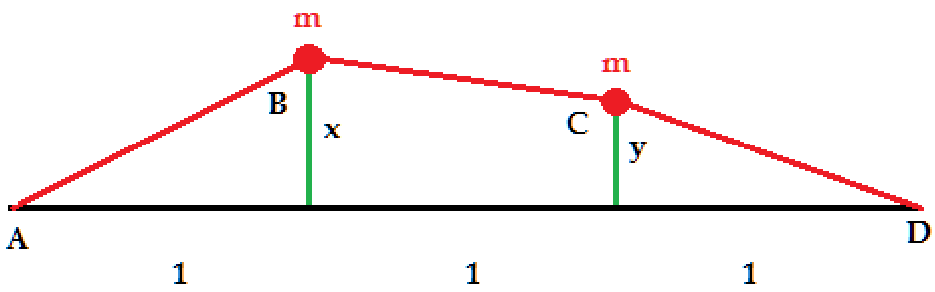

Fig.2

What is the potential function V(x, y) ?
In course lessons and textbooks the potential is normally obtained by the force attributed to Hooke, which I transcribe here in its general form:

$$F = -k\,x$$

By calculating the work done by the force to produce a lengthening x, one obtains the elastic potential :

$$V = \frac{1}{2}kx^2$$

I must say that I don't quite like this method, even if it is frequently used. Landau himself, in his book "Mechanics" deduces (so to speak) Lagrange's Equation by introducing an abstract potential function which, added to the free particle Lagrangian (which is nothing but the kinetic energy of the particle) will allow us to obtain, for example, Newton's equations. Nothing bad in all this, but if this abstract potential function is derived from the Newtonian force, I am left with the impression that "the dog is (at least partially) eating its tail". In other words, in my opinion we should be able to construct the lagrangian L=T-V directly, without invoking Newtonian forces. V should be given either directly by the experiment or by some independent reasoning (indeed, in elementary particle physics there are no other ways). Unfortunately I have against me Hooke himself,who wrote (1678) *"Ut tensio sic vis"*, that is:

$$F = -k\,x$$

An illuminating observation I found (W.W. Sawyer, "A Path to Modern Mathematics" – golden booklet!) is that the potential energy resides in the total lengthening of the elastic string to go from the rest position to the configuration given, for example, in Fig.2. If one stretches the elastic string, the only results of the work which has been done is the lengthening of the same , and therefore we have no better choice, because of a *"principle of sufficient reason", (*dear to the Eighteenth century mathematicians), than to declare that the elastic potential of the stretched string resides in its lengthening form the rest position. The elastic force tents to decrease the length of the string, which has been lengthened by vertically displacing the two masses, to bring it back to the initial length (in our case L=3). The total length after stretching is:

i) For the segment AB: $\sqrt{1+x^2}$, which for x small reduces (series expansion!) to

$$a_1 = 1 + \frac{1}{2}x^2$$

ii) For the segment BC: $\sqrt{1+(x-y)^2}$, which for small displacements reduces to :

$$a_2 = 1 + \frac{1}{2}(x-y)^2 = 1 + \frac{1}{2}x^2 + \frac{1}{2}y^2 - xy$$

iii) For the segment CD: $\sqrt{1+y^2}$, which for small displacements reduces to :

$$a_3 = 1 + \frac{1}{2}y^2$$

The total lengthening (to which the elastic potential will be linked by some constant measured in suitable units)

$$a_1 + a_2 + a_3 - 3 = \boldsymbol{x^2 - x\,y + y^2}.$$

(2) Second problem: motion of a point-mass which slides without friction in a bowl under the effect of gravity.

The shape of the bowl is selected by us. Following Sawyer, we can proceed as follows:

(1) we define a function z (x,y), such as:

$$z = x^2 - \; x\, y + \; y^2$$

(2) we draw two orthogonal axes on the plane, and then, at each point (x,y) raise a straight line segment of height z perpendicular to the plane.

(3) we extend a flexible surface which touches all points, as many as possible, which we have thus constructed.

If we do things well, with a suitable number of points, we obtain a smooth surface, called "parabolic ellipsoid" whose principal axes (pink and light blue in Fig.3) are oriented at 45° with respect to the orthogonal axes (x,y) as their projection on the plane shows. Note the two curves (in fact parabolas) POQ (red and pink) and MON (blue), which are also projected on the principal axes of the ellipsoids. Such parabolas are also oriented along the lines of maximum and minimum slope of the surface.

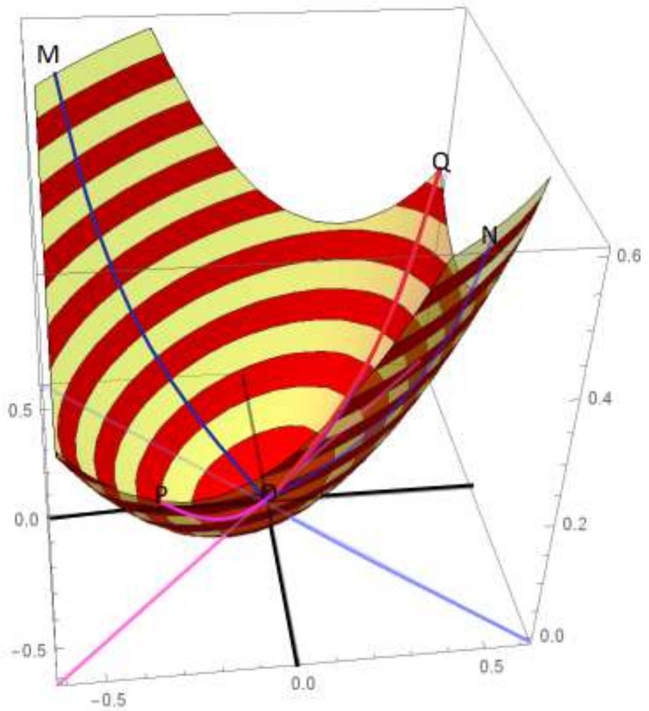

Fig.3

If we cut the parabolic ellipsoid at a height z, and plot the contour curves of the bowl, we have the following figure:

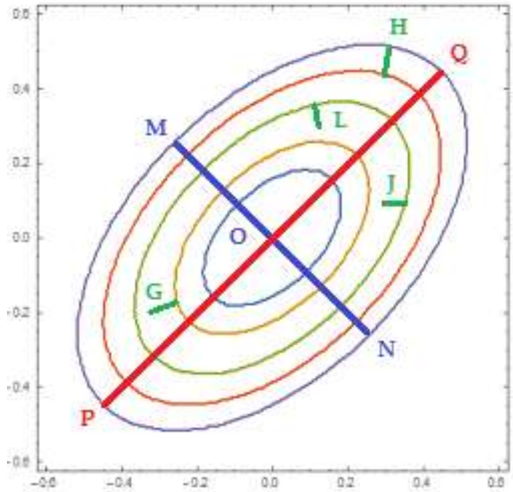

Fig.4

Let's suppose we let go a heavy point from any point, such as H. The particle will (frictionless) slide along the maximum slope lines , which, as we know from a previous essays on the gradient on this same site (http://dainoequinoziale.it/sassolini/2017/07/04/normalegradiente.html) are at any point perpendicular to the contour line going through that point (the contour lines in figure 4 are all ellipses traced in different colors). If the heavy particle starts from H, its trajectory is quite complex, as ti must be perpendicular to the contour lines at each point. The same can be said of the trajectories starting from points G, J, L. In the absence of friction, the heavy particle will raise on the other side of the bowl at the same height of the starting point. It will then come back performing a **complicated oscillatory motion**, but in general will not go back to the starting point H.

WE can see, however, that there are particularly simple two types of motion, along the line MON and along the line POQ. These two trajectories, which are projected onto two straight line segments on the plane (x,y) are at each point perpendicular to the contour curves, and therefore the projections of the motions along them will be two harmonic oscillations on a straight line.

However, we should not expect two equal frequencies, because the MON trajectory is steeper, and therefore faster to be completed than the trajectory along POQ. The period of oscillation along MON is therefore smaller (and the frequency higher) than along POQ.

The usefulness of eigenvalues and eigenvectors is that any oscillatory motion in the ellipsoidal bowl we are considering, no matter how complicated, can be decomposed into two rectilinear oscillatory motions of suitable frequency. Such two particularly simple motions are called the "Normal Modes" of the oscillating system, and behave as the two orthonormal vectors of a bi-dimensional system, in the sense that any othe vector of the system can be expressed in terms of its components along such basis vectors.

The mathematics we have studied should confirm our qualitative remarks.

To see it, we must diagonalize the matrix associated to our quadratic form, find the eigenvalues (which, as we shall see, are linked to the frequencies), and the eigenvectors (which are linked to the nature of the two simplest motions, or Normal Modes), which, as we have now said and repeated, will be used to describe any more complicated motion.

The symmetric matrix which originates our quadratic form is:

$$\begin{pmatrix} \mathbf{1} & \mathbf{-0.5} \\ \mathbf{-0.5} & \mathbf{1} \end{pmatrix}$$

(and this can be demonstrated readily if we perform the multiplication:

$$(\boldsymbol{x} \quad \boldsymbol{y})\begin{pmatrix} 1 & -0.5 \\ -0.5 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

The eigenvalues can be readily computed by assigning the right values to the coefficients of the determinant given in (2) :

$$\begin{cases} (1-\lambda)x - 0.5\,y = 0 \\ -0.5\,x + (1-\lambda)y = 0 \end{cases}$$

Easy calculations show that

$$\lambda_1 = 1.5, \quad \vec{v}_1 = \frac{1}{\sqrt{2}}\begin{pmatrix} -1 \\ 1 \end{pmatrix}$$
$$\lambda_2 = 0.5, \quad \vec{v}_2 = \frac{1}{\sqrt{2}}\begin{pmatrix} -1 \\ -1 \end{pmatrix}$$

We thus see that the larger eigenvalue, which, as we shall see, corresponds to the highest frequency, is oriented from North-West to South-East, with a slope of 45° (along the MON trajectory), and the smaller eigenvalue from North-East to South-West, the POK trajectory, as announced.

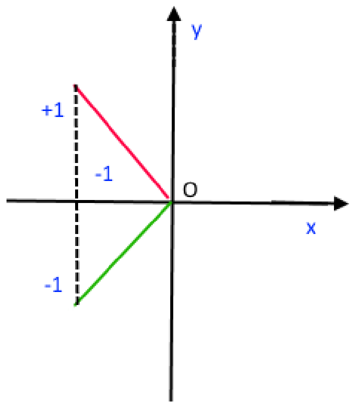

Fig.5

From the values of the components (x,y) of the eigenvectors, we see that they are equal for $\vec{v}_2$ and opposite for $\vec{v}_1$.

If now we abandon for a moment the elliptical bowl, and go back to the oscillating system of Fig.2, where x, y are not the *coordinates* of a single point , but the *ordinates* of two different points, the Mathematics of the two systems being the same, we see that the two eigenvectors, or Normal Modes of the oscillations, have ordinates x, y, which are equal and opposite in the case of the larger eigenvalue, and equal in the case of the smaller eigenvalue.

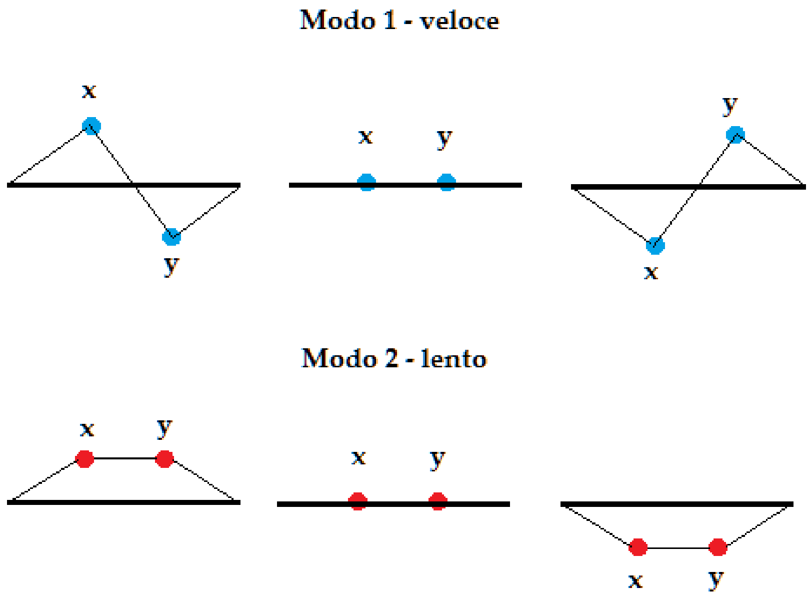

Fig.6

One should not be astonished because the "red" mode has an oscillation frequently smaller that the "blue" mode. In fact, in the blue mode, the two equal mass points for equal |x| and |y|, (excepting the case in which the two masses lie on the rest line) are always on the opposite sides of the rest line, which means that the elastic string joining them is longer, that is more stretched, and the two masses are subjected to greater elastic force, which produces a greater oscillation frequency.

Let's go back to the bowl. To have a general view of the behavior of the oscillations of a heavy particle, we only need to remember how do we arrive at the equations of motion of two coupled oscillators (which, as we have seen from a mathematical point of view are the same problem).
For two coupled oscillators we inevitably arrive at the two equations of motion:

$$\begin{cases} ax + cy = \ddot{x} \\ cx + \ by \ = \ddot{y} \end{cases}$$

To solve these equation one is usually given the vague statement (which I discussed already in
http://dainoequinoziale.it/resources/sassolini/leprimequazioni.pdf) that in a linear, second order system with constant coefficients we can try to find a solution by setting:

$$x = Ae^{i\omega_1 t};\ \ y = Be^{i\omega_2 t}$$

A priori, as I already said in the quoted essay, no reason is given to do so, but at least we know that the method works. With this position we obtain:

$$\begin{cases} ax + cy = -\omega_1^2 x \\ cx + by = -\omega_2^2 y \end{cases}$$

Which, written in different form, is none other than:

$$\begin{cases} ax + cy = \lambda x \\ cx + by = \lambda y \end{cases}$$

In which the eigenvalues we shall find are the squares of the frequencies. The problem, which the minus sign creates can be avoided, as we know, by using trigonometric functions, if we do not like to use complex numbers.

The motion of a point R(u,v) whose u- component oscillates along one of the ellipsis axes, while the v-component oscillates along the other axis, results from the following construction:

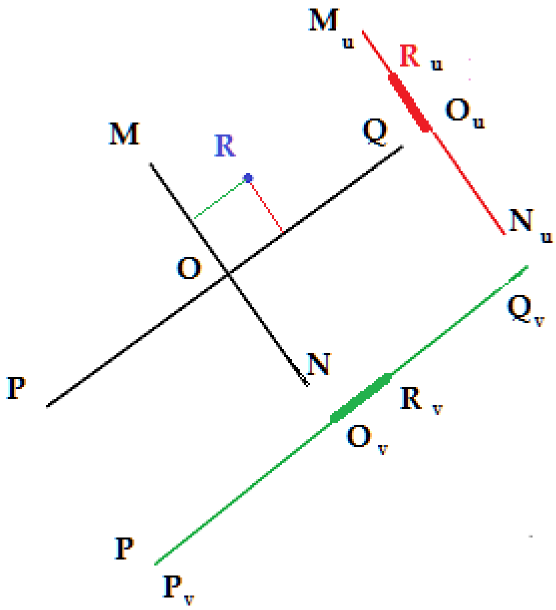

Fig.7

The coordinates of R, namely R_u and R_v, measured on the oblique axes U and v, with oscillation amplitudes equal to ($M_u - N_u$ and $P_v - Q_v$), oscillate with frequencies equal to the square roots of the found eigenvalues, ignoring the minus sign.

The trajectory of R can assume rather complicated figures, which are named **"Lissajous curves (or figures)".**

To give an example of Lissajous curves , using the Wolfram Mathematica language, I hae drawn a rough example of hat happen in our case, when the frequencies assume the values of $\sqrt{1.5}$ (with semiaxis 0.5) and $\sqrt{0.5}$ (with semiaxis 1). As the ratio of the two frequencies is not a rational number, the trajectory will never closed. In fact, little by little it wil fill the rectangle of dimensions 2 x 1, in the sense that point R sooner or later will pass infinitely close to any point of the rectangle.

Using the formula

ParametricPlot[{Sin[Sqrt[0.5]t+ Pi/2.] ,0.5 Sin[Sqrt[1.5]t+ Pi/2.]}, {t, 0,20}]

in 20 time units, starting from K; one obtains the small portion of the trajectory:

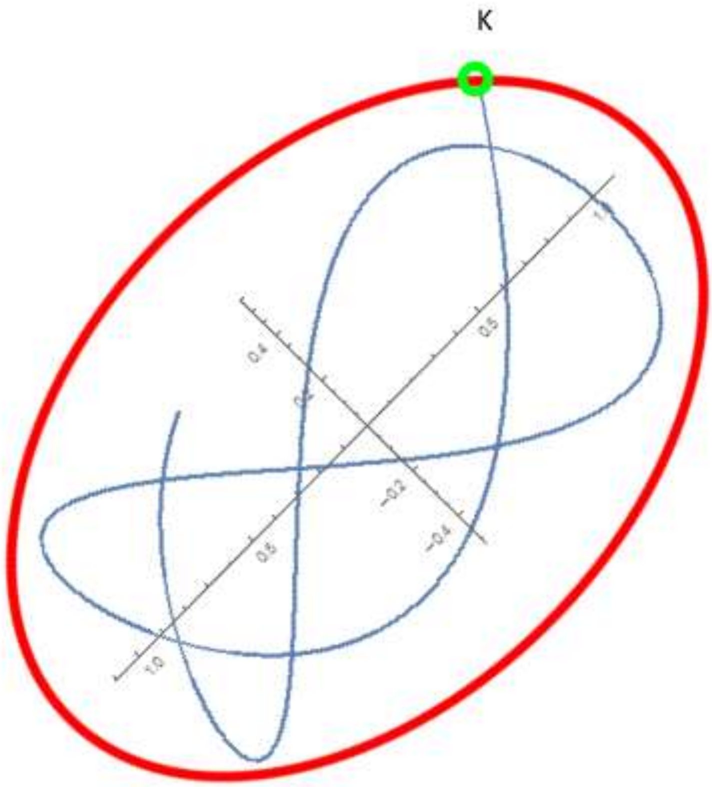

Fig.8

C. The quadratic form $z = x^2 + 2\,x\,y + y^2$.

The associated symmetric matrix is:

$$\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$$

whose determinant is Zero, as the matrix has two equal rows (and two columns) – although in this case doing the calculation by heart is faster.

It is known that the determinant changes sign if two rows are exchanged, and therefore, if there are two equal rows the determinant should change sign and at the same time keep the same value, once they are exchanged. Therefore it must have zero value.

The eigenvalues are $\lambda_1 = 2$, corresponding to the eigenvector (1,1), and $\lambda_2 = 0$, corresponding to the eigenvector (-1,1). Note that the eigenvectors are not normalized, which in this case is unimportant. The interested reader can try to rnormalize them. The two eigenvectors are manifestly orthogonal (and their internal product proves it). The eigenvectors once more point to the two lines at 45 degrees with respect to the axes x and y. If we take the line y = x, we find that along such a line, $z = 4x^2$, that is, we have a parabola. If we take x = -y, we get z=0, a straight line. This explains why we have an indefinite case: the line z = 0 along the straight line y = - x does not allow the origin to be either a minimum or a saddle point, as fig.2 shows.

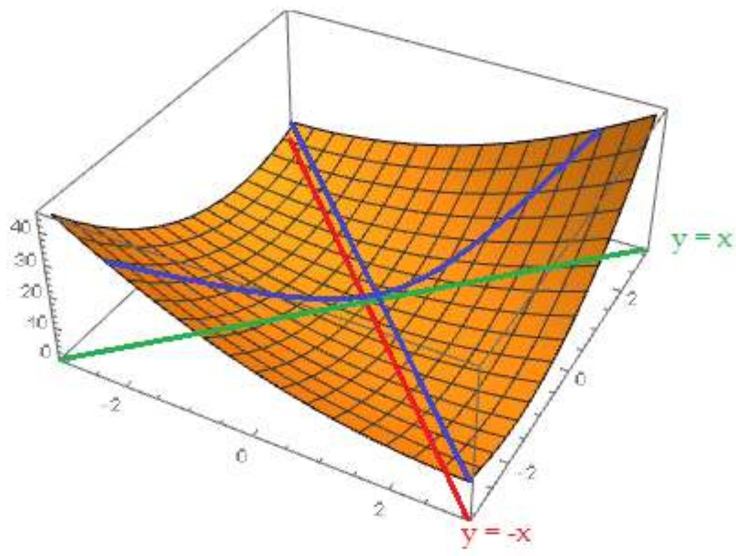

Fig.2 (made using the Wolfram-Mathematica language)

Contrary to what one could believe following careless reasoning, therefore, there may exist points where the derivatives exist, but, as one can see from Fig.2, we cannot have neither a maximum nor a minimum nor a saddle point. Ant the situation might not get better even examining the higher order terms of the Taylor series expansion of suitable functions. In any case, the anomaly we have shown is not the only anomaly which can arise of the Hessian Determinant = 0.

X.

And now I hope you are ready for a surprise. Undoubtedly, the Hessian is an interesting animal in the mathematical zoo, with many applications which I will list in the Conclusions, but *it is not the only mathematical instrument absolutely necessary to reduce a quadratic form to its canonical form.* The demonstration, which follows, can be extended to n variables, but becomes especially simple, inevitably, in two dimensions.

Let's consider the quadratic form

$$Q = ax^2 + 2bxy + cy^2$$

Whose associated matrix is:

$$\begin{pmatrix} a & b \\ b & c \end{pmatrix}$$

Let's work now on the portion of Q which contains x. It is: $ax^2 + 2bxy$.
However, for a ≠ 0:

$$\frac{1}{a}(ax + by)^2 = ax^2 + 2bxy + \frac{b^2}{a}y^2$$

And therefore :

$$Q = \frac{1}{a}(ax + by)^2 - \frac{1}{a}(b\,y)^2 + cy^2 = \frac{1}{a}(ax + by)^2 + \left(c - \frac{b^2}{a}\right)y^2$$

By putting

$$A = \frac{1}{a}$$
$$X = (ax + by)$$
$$B = \left(c - \frac{b^2}{a}\right)$$
$$Y = y$$

One has:

$$Q(X,Y) = AX^2 + BY^2$$

That is, the quadratic in canonical form.

Our original quadratic form we intended to study was:

$$\begin{pmatrix} a & b \\ b & c \end{pmatrix} = \begin{pmatrix} f_{xx} & f_{xy} \\ f_{xy} & f_{yy} \end{pmatrix}$$

Which allows us to assign explicit values to the coefficients and to the variables):

$$A = \frac{1}{f_{xx}}, \quad B = \left(\frac{f_{xx}f_{yy} - f_{xy}^2}{f_{xx}}\right), \quad X = (x\, f_{xx} + y\, f_{xy}), \quad Y = y$$

The Quadratic form becomes:

$$Q(X,Y) = \frac{1}{f_{xx}}\,(x\, f_{xx} + y\, f_{xy})^2 + \left(\frac{f_{xx}f_{yy} - f_{xy}^2}{f_{xx}}\right)Y^2$$

Clearly, the coefficient B is the Hessian determinant divided by f_{xx}. We see once more that:

1. If D>0 and $f_{xx} > 0$, Q is larger than 0, and we have a minimum of f(x,y);
2. If D>0 and $f_{xx} < 0$, Q is smaller than 0 and we have a maximum.

For the case D < 0, we must examine the whole quadratic form.
We know that its Hessian is negative. If f_{xx} is negative, we have that the first term is negative and the second positive; if f_{xx} is positive we have the opposite.
1 Case: $f_{xx} > 0$.
Putting Y (=y)= 0, one gets that along the x axis we have the positive parabola Q(x) = $f_{xx}\, x^2$, while, along the y axis , putting $X = 0$, we have a negative parabola, because such is the sign of the coefficient of Y^2. Note that setting X =0 means that :

$$\frac{x}{y} = -\frac{f_{xy}}{f_{xx}}$$

Which can be done, as f_{xx} is not zero.

2 Case: $f_{xx} < 0$

Proceeding as in the previous case, it is immediate to verify that the signs of the two parabolas are exchanged.

Thus, in both cases we have a saddle-point, as Fig.3 shows for $f_{xx} > 0$, considering the two orthogonal parabolas of opposite sign, going through the critical point (in this case the origin):

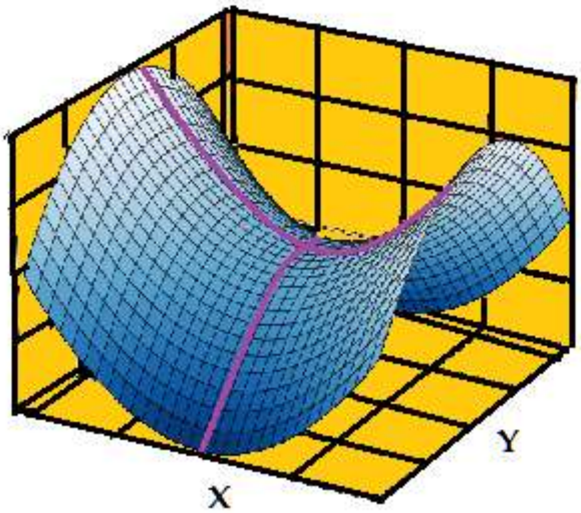

Fig.3 (1 Caso: $f_{xx} > 0$)

Adapted from:
https://upload.wikimedia.org/wikipedia/commons/1/1e/Saddle_point.svg
By Nicoguaro (Own work) [CC BY 3.0 (http://creativecommons.org/licenses/by/3.0)], via Wikimedia Commons

This method, in two variables, is immediately understandable, and brings to the same conclusions as the use of the Hessian. More important, the fact that the numerator of B is indeed the Hessian determinant, gave me the suspicion that Hesse started from here to develop his theory: in a sense the Hessian fell into his hands from heaven, while he was analyzing general quadratic forms, and on its basis he developed his theory regarding bi-dimensional surfaces.

CONCLUSIONS.

The Hessian (be it the matrix or the determinant) is stuff for students who have already gone beyond the first steps in Mathematics. A bit of wordiness, on my part, deriving from my experience with students and their most common doubts, together with a fairly large font, have made a short essay a monster of 35 pages. The question which always arises in the mind of students is "Is it worth it?".

Well, quadratic forms enter at least in the following fields:

1. Number theory,
2. linear algebra,
2. group theory(orthogonal group),
3. differential geometry (Riemannian metric, in particular the "second fundamental form"),
4. differential topology (intersection forms of four-manifolds; Morse theory),
5. Lie theory (the Killing form),
6. Catastrophe theory….

If not a single one of these fields is of your interest, then you should think that Math is about such (and similar) objects, and draw your conclusions.